Concepts in Theoretical Physics

Lecture 2: Invariances, Constants and Dimensions

John D Barrow

'I consider that I understand an equation when I can predict
the properties of its solutions without actually solving it.'
Paul Dirac

Trends in the history of physics

- Notice common causes and effects in Nature
- Propose laws of change
- Notice that they imply some things never change
- Constants and conserved quantities
- Frame the laws as invariances (no special observers)
- Derive the laws from the requirement of invariance using tensor analysis and group theory
- This is the basis of fundamental (mathematical) physics

Invariances

- Conserved quantities $\leftrightarrow$ invariances of laws/eqns
- Same at all times, in all places and in all directions $\rightarrow$ energy, momentum and angular momentum are conserved.
- Newton's F = mA is anti-Copernican 'undemocratic'. Only seen by special observers who are not accelerating or rotating (with angular velocity $\Omega(t)$.
- In general

$$\mathbf{F'} = m\mathbf{A} - 2m\boldsymbol{\Omega}\wedge\mathbf{v} - m\boldsymbol{\Omega} \wedge(\boldsymbol{\Omega} \wedge \mathbf{x}) - m(d\boldsymbol{\Omega}/dt) \wedge \mathbf{x}$$

angular velocity vector $\boldsymbol{\Omega}$ points along the axis of rotation, the body is at position $\mathbf{x}$ with velocity $\mathbf{v}$ as seen by a rotating observer. The extra force terms are the Coriolis + centrifugal + Euler 'forces' and they differ in every rotating frame with different $\boldsymbol{\Omega}$.

Newton’s Laws are un-Copernican

Newton’s laws only hold for special observers
who do not rotate or
Accelerate with respect to the fixed stars

There are special classes of observers for whom the laws of Nature look simpler!!

Einstein Extends the Copernican Principle from Outcomes to Laws

The forms of the laws of nature look the same to ALL observers regardless of their motion

Tensors: $T = S \rightarrow T' = S'$ as $x \rightarrow x'$

Tensors are just what you need

- In one coord system, x^a : if we have eqns $A_{cd} = B_{cd}$
- Changing to any set of new coords $x^{a\prime}$ moving arbitrarily wrt the x^a , laws will have the same form: $A_{cd}{}' = B_{cd}{}'$ in the new coords if A and B both transform as tensors:

$$A'_{cd} = \frac{\partial x^f}{\partial x'^c} \frac{\partial x^g}{\partial x'^d} A_{fg}$$

with the same rule for $B_{cd}{}' \rightarrow B_{fg}$ and repeated indices are summed over here.

Express the laws of physics in tensor form

Constants of Nature

- In the equations of physics the constants of proportionality matter ($E = mc^2$)
- They are related to fundamental defining quantities of the universe, like G, c, h, m_e, ..
- The so called ‘standard model of particle physics based, on the symmetries $SU(3) \times SU(2) \times U(1)$ has 19 constant parameters. To which you add gravitational constants, like G, and cosmological parameters, k, Λ, and initial conditions. They may all be linked?!
- We have not explained or predicted the values of any constants of Nature.

Einstein's Classification

1. Geometrical constants, e, 2π, etc
2. Constants reflecting units 'apparent constants'
3. Dimensionless parameters 'real constants' (ultimately reducible to constants of type 1
4. Possible cosmological parameters

'Dimensionless constants in the laws of nature, which from the the purely logical point of view can just as well have different values, should not exist.' A. Einstein

Today: uniqueness of constants of Nature ↔ Unique vacuum state

No reason to believe in this uniqueness (string theory….)

Constants of order unity are usually small

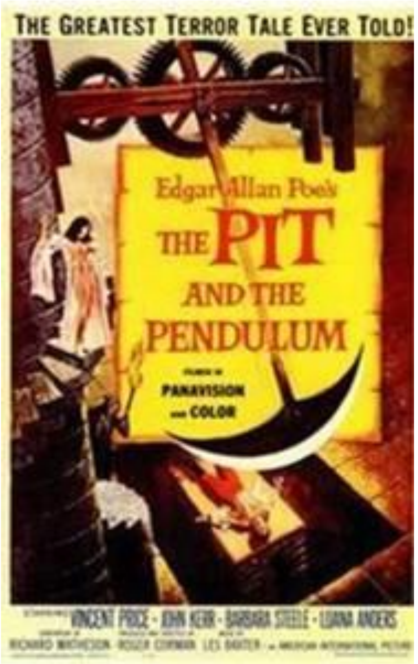
Simple pendulum of length L has period T = 2π√(L/g).

Why is the "2π" factor so often O(1) in physics?

Why does dimensional analysis work so well?

Einstein's was puzzled by this question in 1911.

Here's my answer. The volume and surface area of N dim spheres are (check N=3,2,1):



$$V_N(R) = \frac{\pi^{N/2}}{\Gamma\left(1+\frac{N}{2}\right)} R^N,$$

$$A_N(R) = \frac{2\pi^{(N+1)/2}}{\Gamma\left(\frac{1+N}{2}\right)} R^N,$$

where $\Gamma(..)$ is the gamma function. We see that if the dimension of the world was much larger than $N = 3$ that the dimensionless factor in from of R^N would depart greatly from being $O(1)$ and dimensional analysis might not be so useful

Mathematicians hate units

- There are 1000's of them: SI, cgs, solar masses, astronomical units, bits, bytes, days amps, watts, slugs, horsepower, SWG, gross, dozens, radians, degrees, stones, furlongs, miles, Mach numbers, Fahrenheit and Centigrade,…. Many were generated by the industrial revolution and continue to be created today (bytes, gigaflops, etc). They're so 19th century!
- They are motivated by human scales (feet and kilograms) or terrestrial motions (days and years) and defined by reference to standards
- Can we escape anthropomorphic biases? We can use atomic standards but what about when there were no atoms?

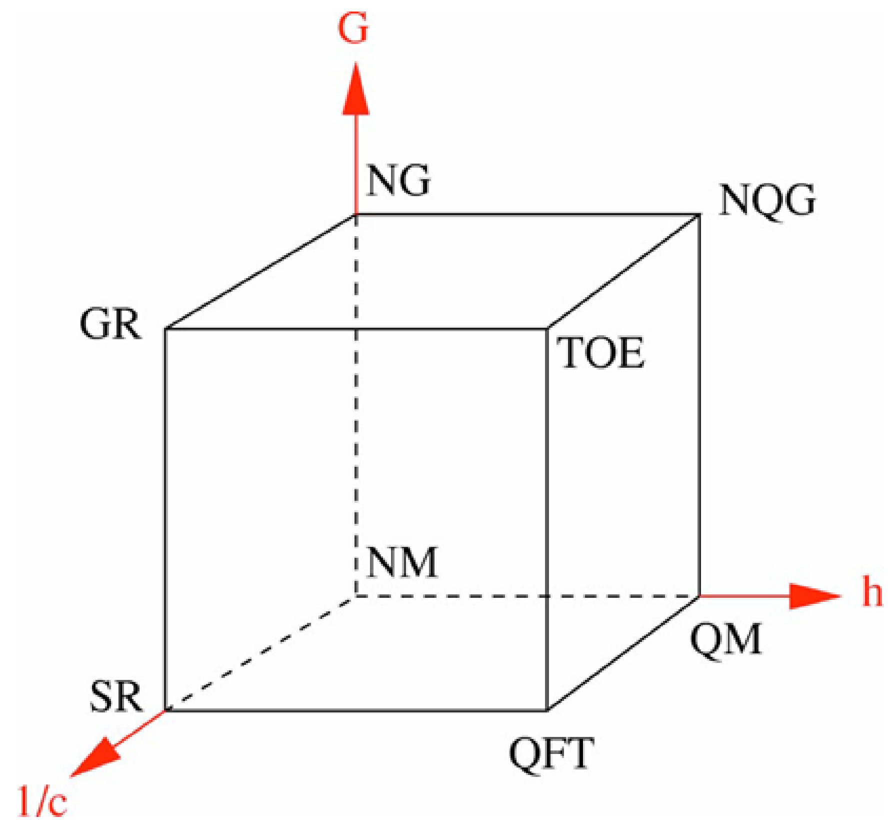
Max Planck's Natural Units (1899)

- Use dimensional analysis to make units of mass length and time from G, h and c, the constant of gravitation, Planck's constant and the speed of light.
- $[G] = M^{-1} L^3 T^{-2}$, $[c] = L T^{-1}$, $[h] = M L^2 T^{-1}$
- The only combinations of G, c, h giving a mass, length and time are:
- $l_{pl} = (Gh/c^3)^{1/2} = 4 \times 10^{-33}$ cm
- $t_{pl} = (Gh/c^5)^{1/2} = 10^{-43}$ s
- $m_{pl} = (hc/G)^{1/2} = 6 \times 10^{-5}$ gm
- The expansion age of the universe is about $10^{60}\, t_{pl} \approx 13.7$ Gyr
- This idea of 'Natural Units' was first created by the Irish physicist George Johnson Stoney in 1874. He used c, G and e (the constant that he created and whose value he predicted before the electron – which he named -- was discovered).

Maximum force conjectures

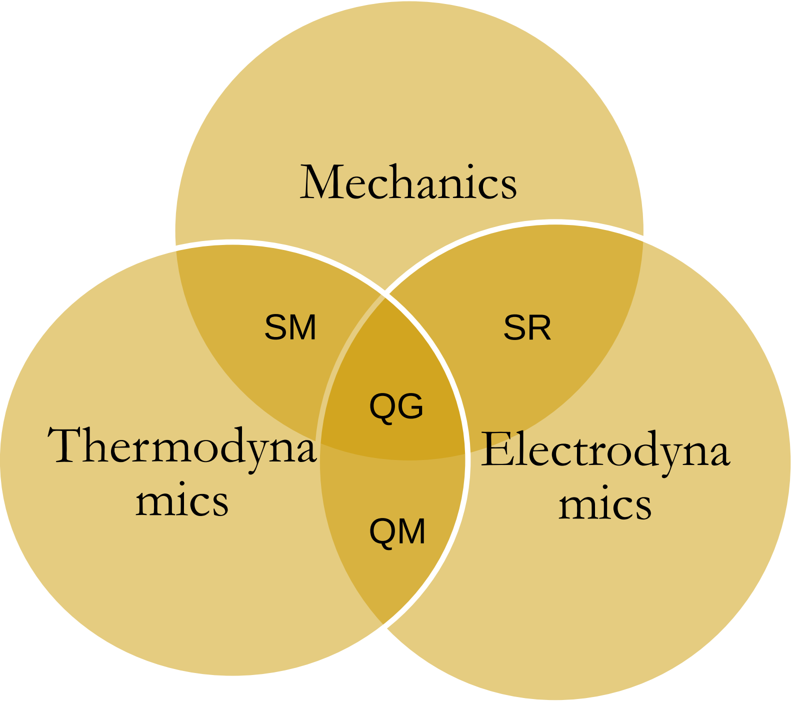
- Forming Planck units can yield new fundamental insights when they are non-quantum
- Natural unit of force $F_{pl} = c^4/G$ does not contain h
- F_{pl} (or ¼ F_{pl}) may be a maximum force in Nature just as the speed of light, c, is a maximum speed.
- This is not true in Newtonian gravity where F is unbounded because $F \propto 1/r^2 \rightarrow \infty$ as $r \rightarrow 0$ for point-like particles. But black hole formation in general relativity stops this physical infinity occurring! We will return to this in a later lecture.

The Cube of Physics from G, c and h



NG = Newtonian gravity, NQG = Newtonian quantum gravity, GR = general relativity, NM = Newtonian mechanics, SR = special relativity
QM = quantum mechanics, QFT= quantum field theory, TOE = 'theory of everything'

Interfaces



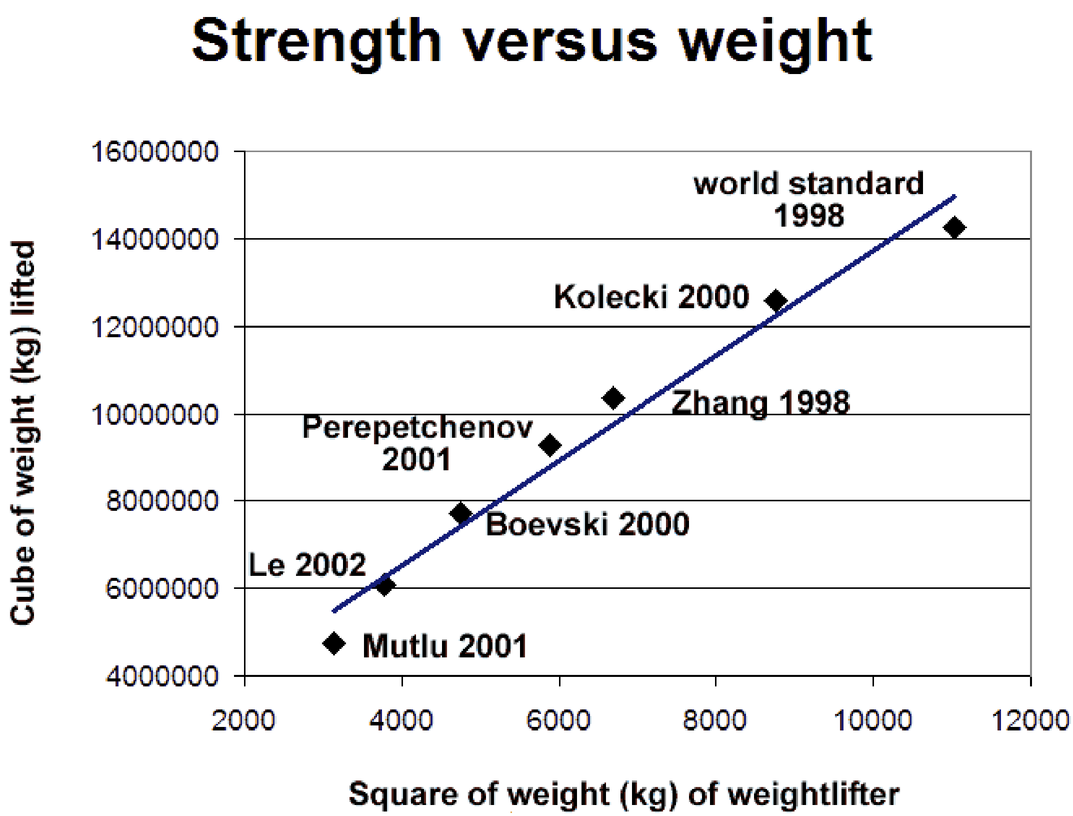
SM = statistical mechanics; SR = special relativity;
QM = quantum mechanics; QG = quantum gravity

Strength and Weight

strength α area α $(\text{weight})^{2/3}$

Giants eventually break!

Weightlifting records versus lifter's weight



$$(\text{strength})^3 \propto (\text{weight})^2$$

Buckingham's 'Pi' Theorem of Dimensional Analysis

Edgar Buckingham
(1867–1940)

So, returning to the general case, assume that we list the n variables in such a way that $a_1, a_2, \ldots, a_j$ are dimensionally independent, while the remaining ones, $a_{j+1}, \ldots, a_n$, can be written as combinations of the others. Having done this, the π quantities are constructed as follows. Take a_{j+i} and divide it by a product of powers of $a_1, \ldots, a_j$:

$$\pi_i = \frac{a_{j+i}}{a_1^{q_i} a_2^{r_i} \ldots a_j^{z_i}}, \quad for \quad i = 1, \ldots, n-j.$$

The powers $q_i, r_i, \ldots, z_i$ must then be chosen so that π_i has no dimensions. This is Buckingham's 'Pi theorem'. And in case you are thinking it's obvious – it is! But it's very useful.

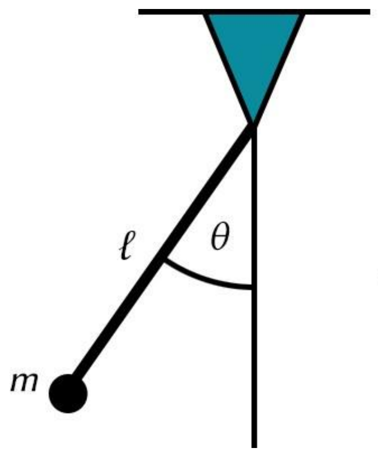
This is the formal statement of the basis of dimensional analysis

A simple (harmonic) example

For example, for the simple harmonic oscillator with length l, bob mass m, at angle θ, swinging with period t under acceleration due to gravity, g, there are $n = 5$ variables on which the motion might depend,

$$[m] = M, [l] = L, \theta = \text{dimensionless}, [t] = T \text{ and } [g] = LT^{-2}$$

Only three are independent so the number of $\pi's$ we need is $5 - 3 = 2$.One is obvious (θ) because it is a dimensionless quantity already, the other is just the dimensionless combination of gt^2/l, so



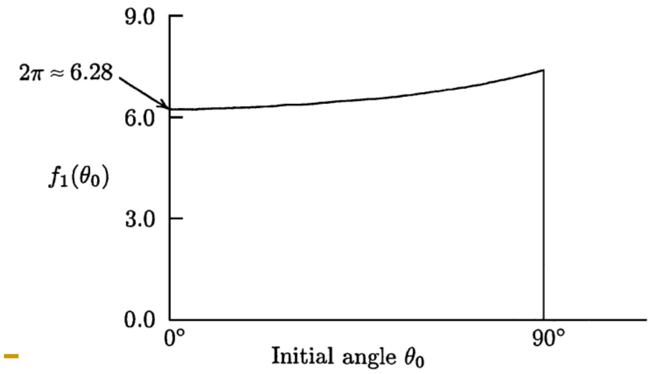
$$\begin{aligned} \pi_1 &= \theta \text{ and } \pi_2 = gt^2/l \\ F(\theta, gt^2/l) &= 0 \end{aligned}$$

From the study of small ($\sin\theta \simeq \theta$) oscillations we know this function $F(..)$ to be

$$\theta = \theta_{\max} \sin\left(\sqrt{\frac{gt^2}{l}}\right)$$

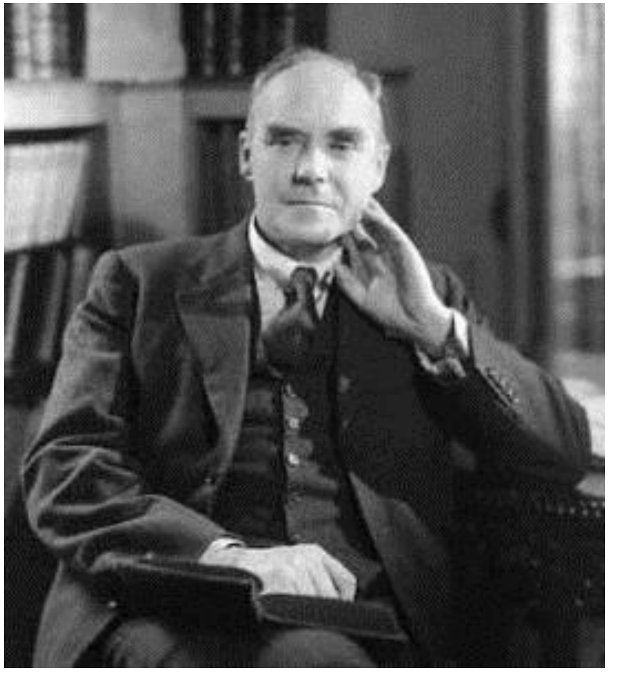
For all θ sizes you could show that the period T of a complete swing is four times the time $t(\theta_0)$ to go from $t = 0$ to θ_0 is (hint – use the conservation of total energy: $mgl(1 - \cos\theta) + \frac{1}{2}ml^2\dot{\theta}^2 = \text{constant} = mgl(1 - \cos\theta_0)$)

$$T = 4\int_0^{t(\theta_0)} dt = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0} \frac{d\theta}{(\cos\theta - \cos\theta_0)^{1/2}} = f_1(\theta_0)\sqrt{\frac{l}{g}}$$



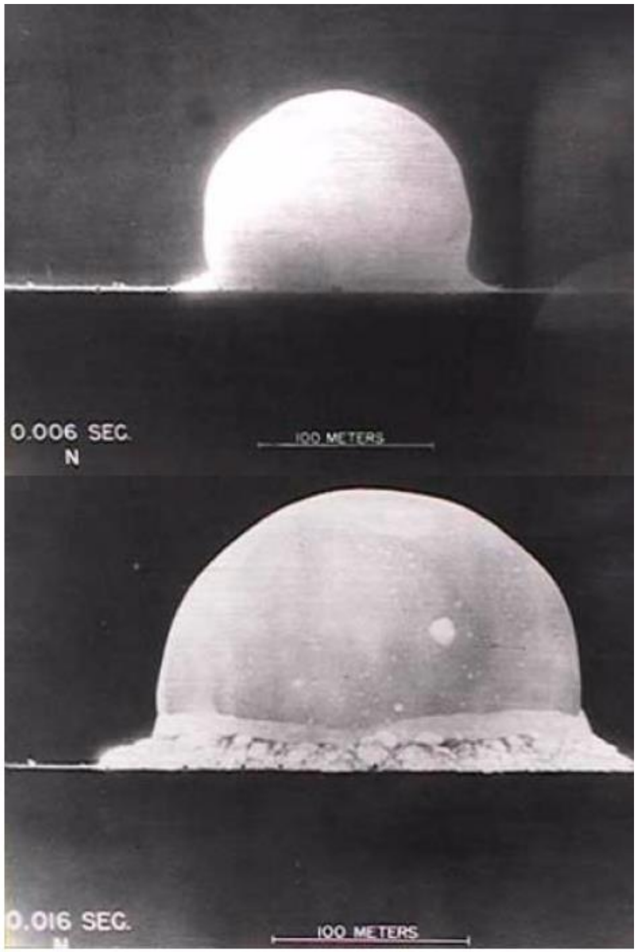
A plot of the elliptic function $f_1(\theta)$ is shown.

Geoffrey Ingram ('GI') Taylor (1886-1975)



He was a professor at DAMTP and his portrait hangs on the
wall of the Pavilion H in the CMS building.
You can see it from the CMS lunch queue

G.I. Taylor and the atomic bomb



In 1947, the US military released Mack's movie of the first atomic bomb explosion, which took place in the New Mexico desert in 1945, under the title 'Semi-popular motion picture record of the Trinity explosion'. Taylor was able to determine, directly from the movie, the relation between the radius of the shock wave and time. To the annoyance of the US government, he was able to work out the energy E released in the explosion and found it to be about 10^{14} J, equivalent to the explosion of about 20 kilotons of TNT. This work was published only in 1950, nine years after he had carried out his first calculations.

What are the relevant variables?

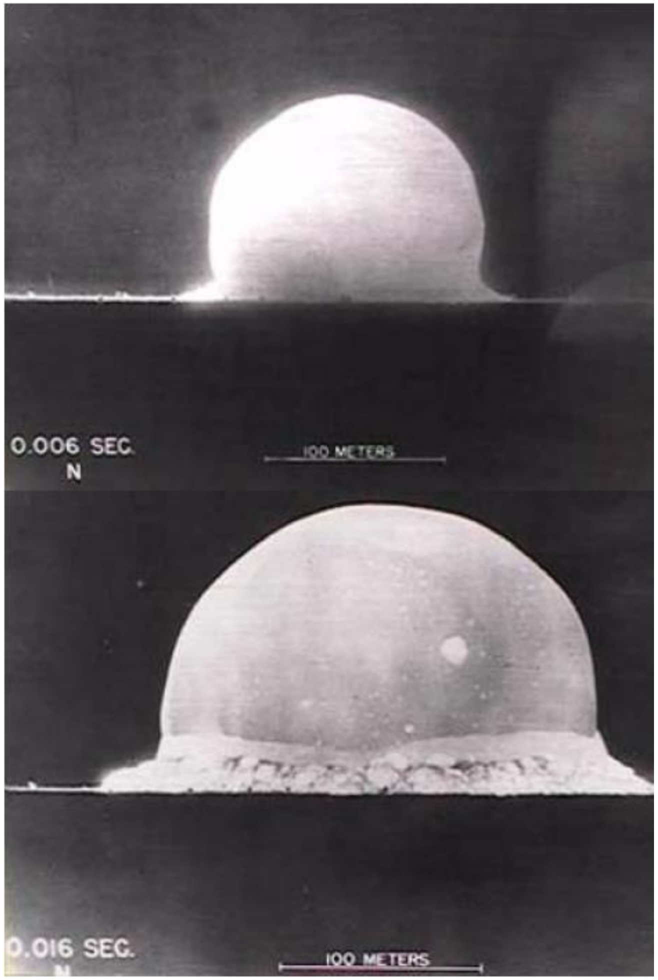
- r, the radius of the shock front.
- ρ, density of surrounding air.
- E, energy released by the device.
- t, the time at which the front reaches r.

4 variables, 3 independent
So there is only 4 - 3 = 1
'pi' quantity, π_1

Deducing the blast-wave formula

- Dimensions: $r = [L]$, $\rho = [ML^{-3}]$, $E = [ML^2T^{-2}]$, $t = [T]$
- $r = C\,\rho^x\,E^y\,t^z$ with C constant
- This requires $x = -1/5 = -y$ and $z = 2/5$
- So $\pi_1 = r^{-5}E\,\rho^{-1}\,t^2$ is the only dimensionless group
- $E = C''\,r^5\,\rho\,t^{-2}$ with C'' a new constant
- Taylor had experimental data indicating that $C'' = 1.033$ for air at the relevant temperature

Estimating the energy yield



In 1947 a movie of the Trinity test explosion was released to the public. In one frame $r = 100$ m at a time of $t = 0.016$ s after the explosion. $\rho \approx 1.1$ kg/m^3 at that altitude.

http://nuclearweaponarchive.org/Usa/Tests/Trinity.html

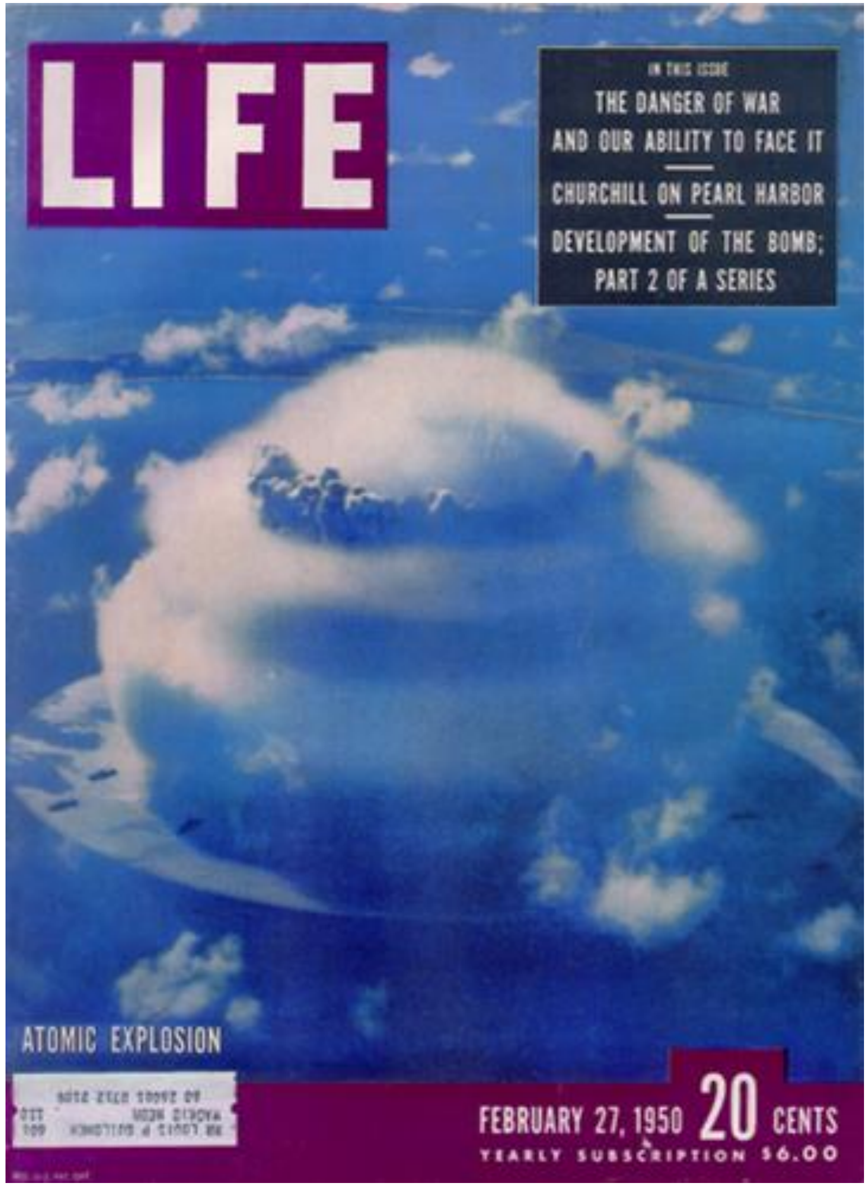
Substitute in these values:

$$E \approx 4 \times 10^{13} \text{ J}.$$

1000 tons of TNT (a kiloton) releases about 4.2×10^{12} J. So the above value is about 10 kilotons TNT equivalent. The actual yield was 18–22 kilotons.

Even closer values can be obtained from other frames. See

http://en.wikipedia.org/wiki/Nuclear_weapon_yield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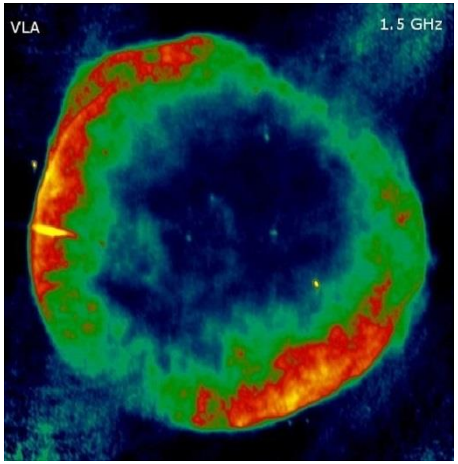

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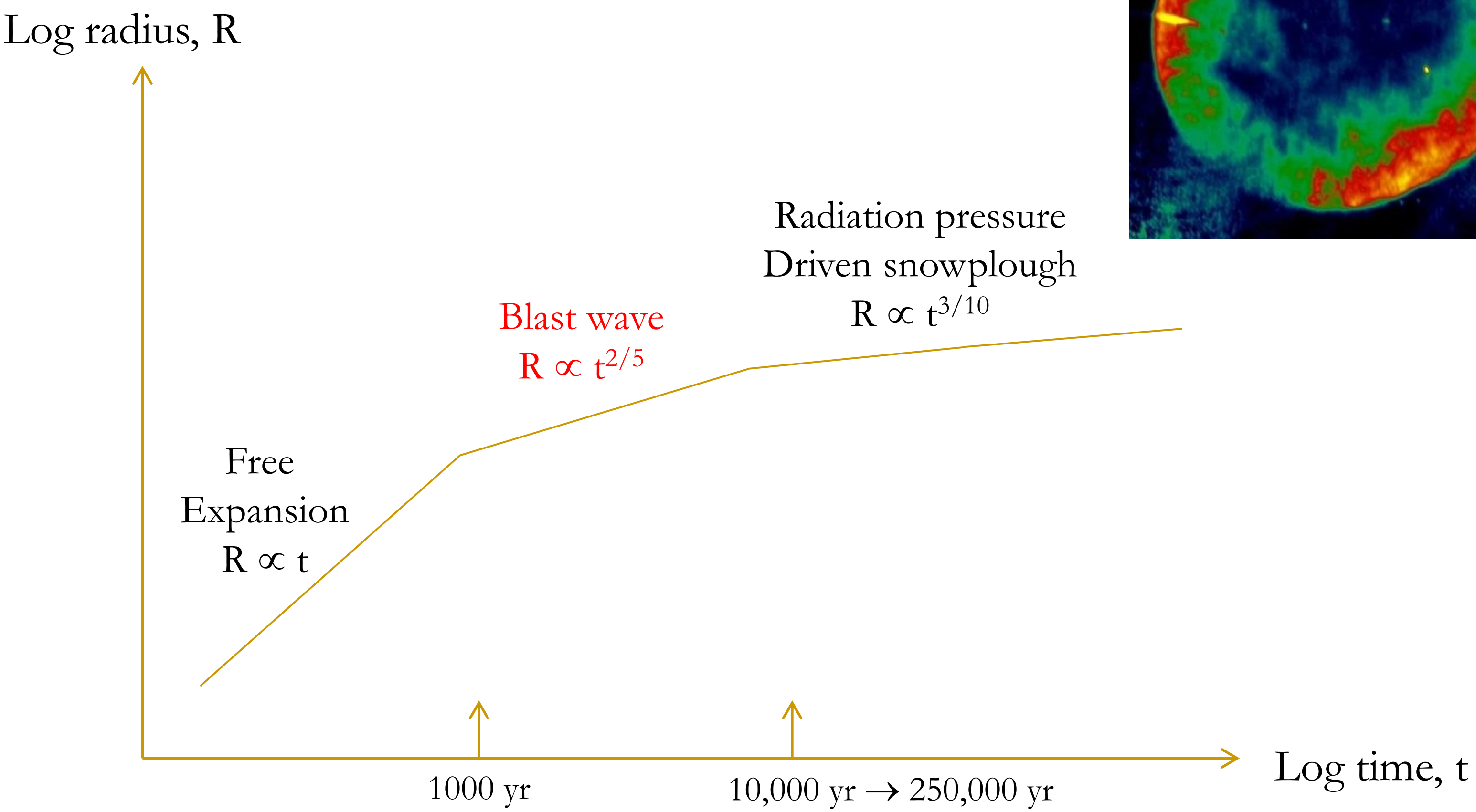

Image: CEA/DSM/DAPNIA/SAp and ESA

Further reading

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- K. Thorne and R. Blandford, *Modern Classical Physics*, Princeton UP, 2017, pp. 908-916. This huge new book covers all of non-quantum physics in more than 1500 beautifully designed pages. Based on graduate courses at Cal Tech.