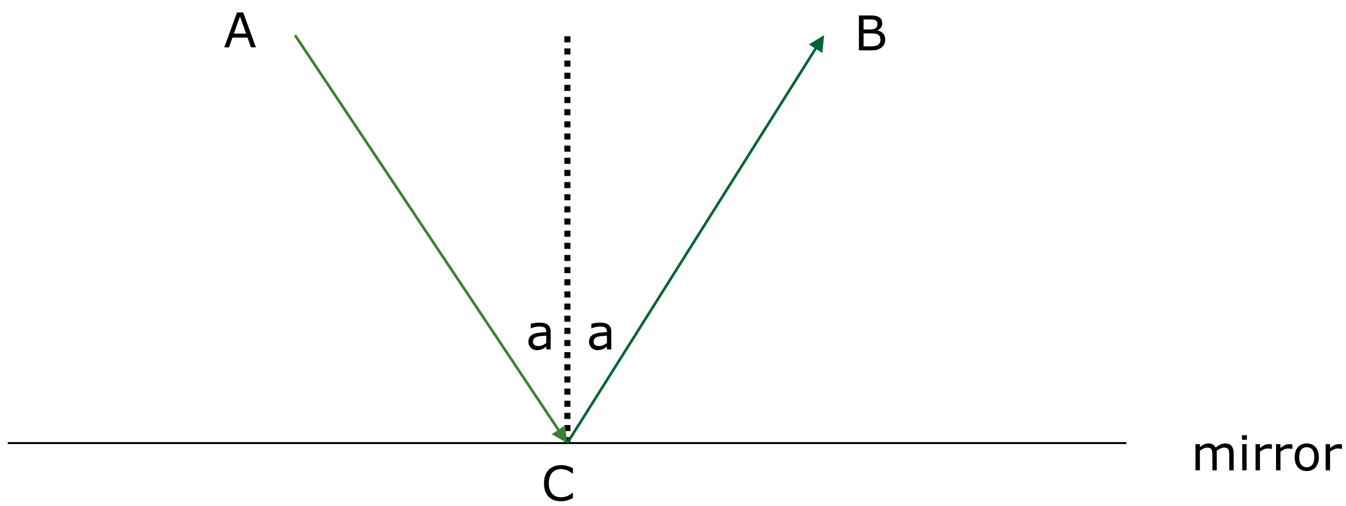
Concepts in Theoretical Physics

Lecture 3: Action Principles

John D Barrow

Actions speak louder than words'
Gersham Bulkeley, 1692

Reflection of Light

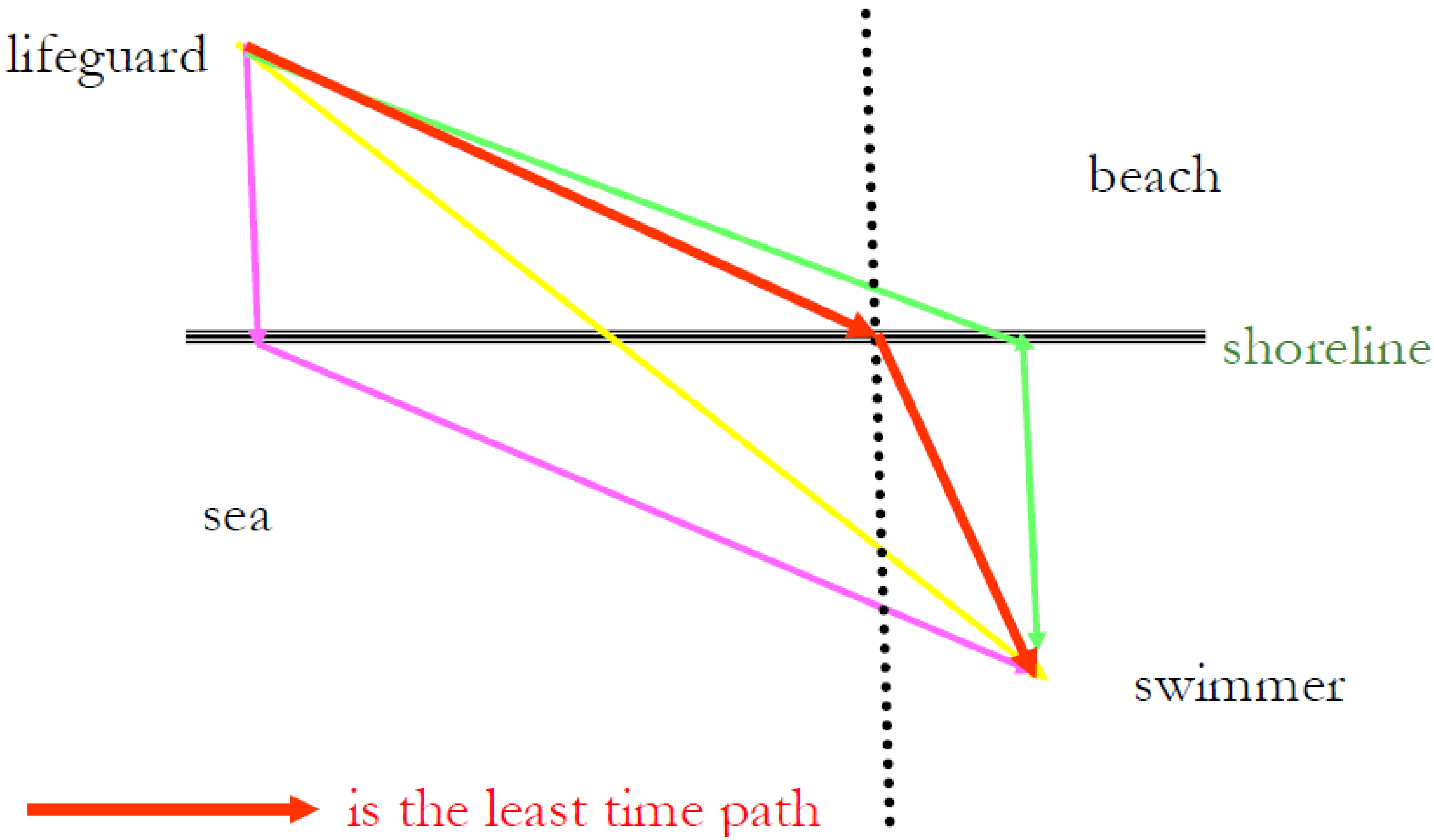


Motion minimises the time to traverse A $\rightarrow$ C $\rightarrow$ B

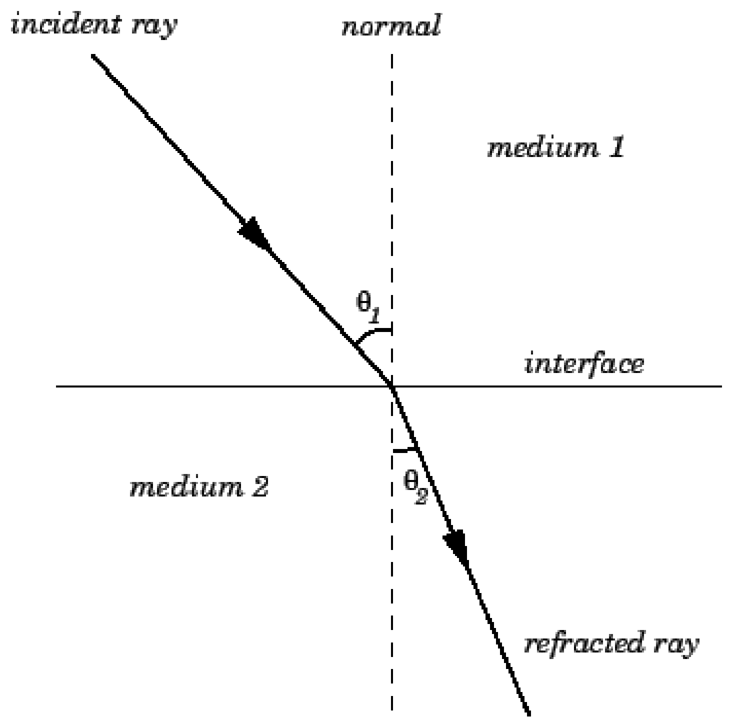
Heron of Alexandria, 1st century BC

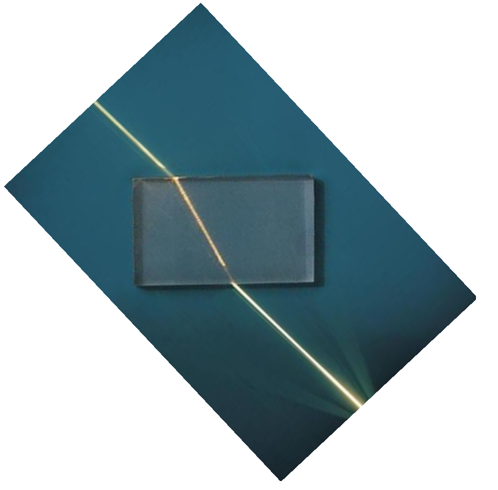
He thought the speed was infinite

The Least Time Path



The Refraction of Light – Snell's Law



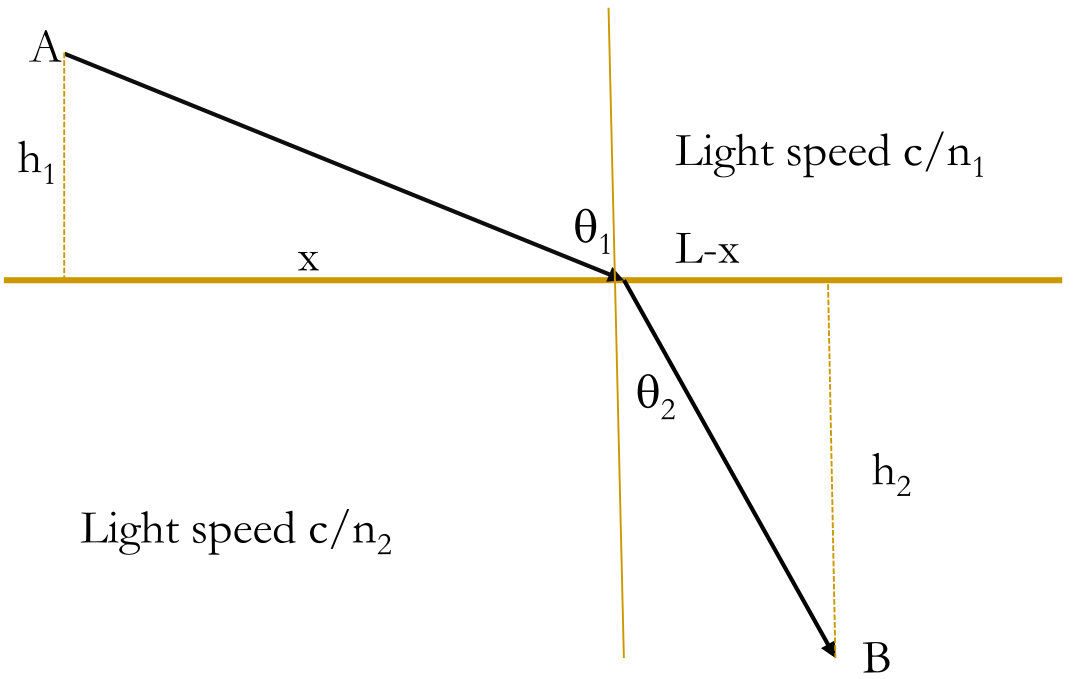


$n = \sin(\theta_1)/\sin(\theta_2)$ = (speed in medium 1)/(speed in medium 2)

n is called the 'refractive index'

→ is the least time path for the light ray

Refraction minimizes light-travel time



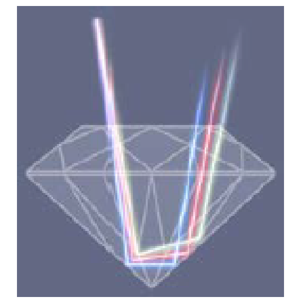
$$T(A\rightarrow B) = \{(x^2 + h_1^{\,2})^{1/2} \div c/n_1\} + \{([L-x]^2 + h_2^{\,2})^{1/2} \div c/n_2\}$$

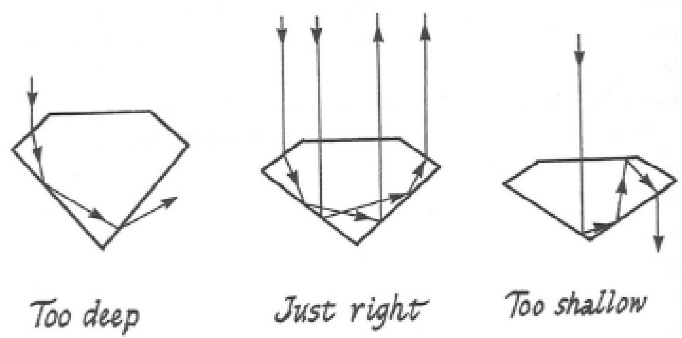
$$\text{Min time: } dT/dx = 0 \rightarrow n_1 x(x^2 + h_1^{\,2})^{-1/2} = n_2(L-x)([L-x]^2 + h_2^{\,2})^{-1/2}$$

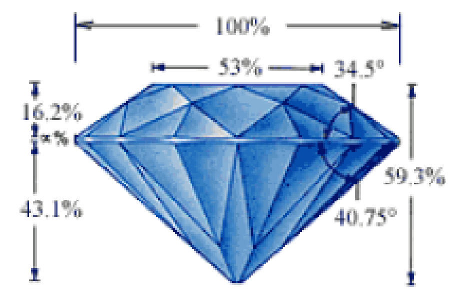
$$n_1 \sin\theta_1 \;=\; n_2 \sin\theta_2$$

Diamond cutting

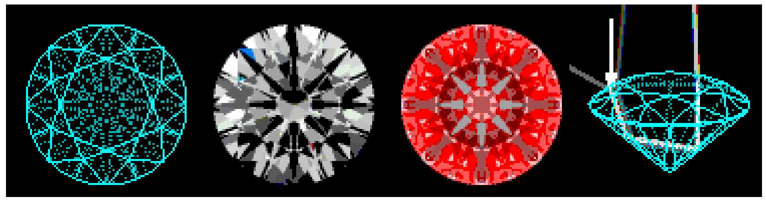
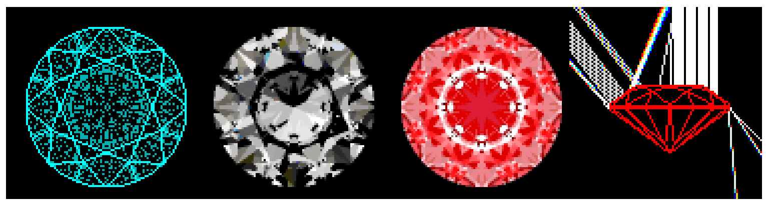
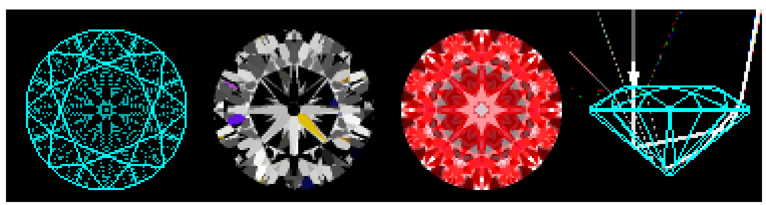
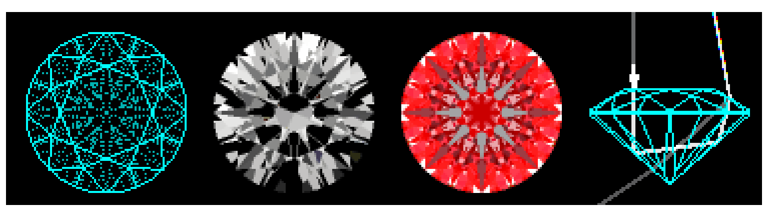
- "Light travels through space, a vacuum, at 3.0 × 10^8 m/s (186,282 miles/s) ie the base refractive index of 1.00,
- when that light hits diamond its speed falls to 1.24 × 10^8 m/s (77,056 miles/s). So diamond has a refractive index of 2.4175.
- Diamond's high refractive index gives very high optical dispersion ('fire') – the variation of refractive index with frequency (colour)

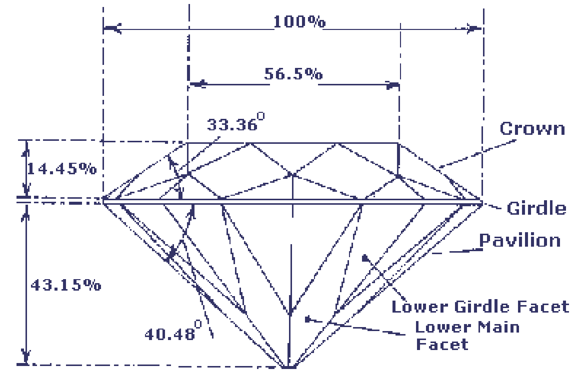






Varying Crown and Pavilion Angles

	Wireframe model of diamond with reflected edges	Photorealistic movies. "Office" - illumination	Photorealistic movies. IdealScope - illumination	Ray Tracing
Crown angle varies from 30 to 40. Pavilion angle 40.7.				
Crown angle 35. Pavilion angle varies from 36 to 46.				
Crown angle varies from 31.25 to 37.75. Pavilion angle varies from 39.15 to 41.85. Direct dependence.				
Crown angle varies from 31.2 to 37.8. Pavilion angle varies from 41.25 to 39.75. Inverse dependence (OctoNus-MSU line).				



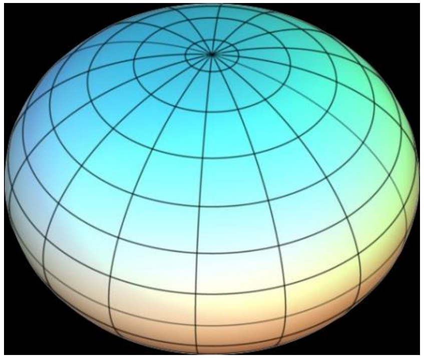
OctNus: Sergey Sivovolenko, Yuri Shelementiev (2002)

Pierre-Louis de Maupertius (1698-1759)

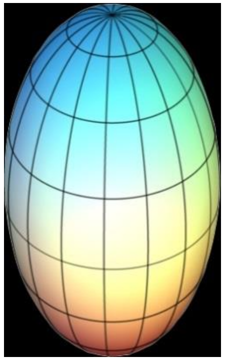
- 1741-6: minimum principles for the motion of masses and the refraction of light
- A principle of ‘economy’ in the construction of the universe
- Fermat, Maupertuis, Euler, Lagrange, and Hamilton
- Maupertuis wanted to counter critics of Leibniz’s ‘best of all possible worlds’ philosophy by providing examples of other ‘possible worlds’ and a definition of what was meant by ‘best’. So he invented the Least Action Principle to silence the critics.

The Earth is Oblate

- In 1736 Maupertuis led an expedition, sent by Louis XV, to Lapland to confirm the Earth was an oblate spheroid as Newton's theory of gravity implied – not prolate as Jacques Cassini claimed from astronomical measurements.



yes



no

Newtonian Mechanics – the old way

- Newton's equation for a single particle with position $\vec{r}$, acted upon by a force $\vec{F}$ is

$$\vec{F} = m\vec{a} \equiv m\ddot{\vec{r}}$$

- The goal of classical mechanics is to solve this differential equation for different forces: gravity, electromagnetism, friction, etc…

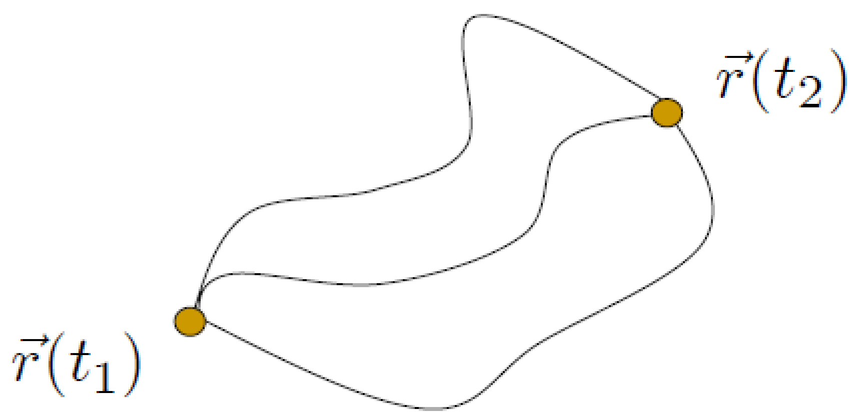
- *Conservative* forces are special. They can be expressed as in terms of a potential $V(\vec{r})$

$$\vec{F} = -\nabla V$$

- The potential depends on $\vec{r}$, but not $\dot{\vec{r}}$. This includes the forces of gravity and electrostatics. But not friction forces.

A New Perspective on Motion

- Instead of specifying the initial position and momentum, let's instead choose to specify the initial and final positions:



- Question: What path does the particle take?

Defining the 'Action'

- To each path, we assign a number which we call the *action*

$$S[\vec{r}(t)] = \int_{t_1}^{t_2} dt \, \left(\frac{1}{2} m \dot{\vec{r}}^2 - V(\vec{r}) \right)$$

- This is the difference between the kinetic energy and the potential energy, integrated over the path. We can now state the main result:

- **Claim:** The true path taken by the particle is an extremum of S.

$$\delta S = 0$$

A Proof of this…

- **<u>Proof:</u>** You know how to find the extremum of a function --- you differentiate and set it equal to zero. But this is a *functional*: it is a function of a function. And that makes it a slightly different problem. You'll learn how to solve problems of this type in next year's "methods" course. These problems go under the name of *calculus of variations.*

- To solve our problem, consider a given path $\vec{r}(t)$. We ask how the action changes when we change the path slightly

$$\vec{r}(t) \to \vec{r}(t) + \delta\vec{r}(t)$$

- such that we keep the end points of the path fixed

$$\delta\vec{r}(t_1) = \delta\vec{r}(t_2) = 0$$

The Proof continues…

$$S[\vec{r}+\delta\vec{r}] \;=\; \int_{t_1}^{t_2} dt\,\left[\frac{1}{2}m(\dot{\vec{r}}^{\,2}+2\dot{\vec{r}}\cdot\delta\dot{\vec{r}}+\delta\dot{\vec{r}}^{\,2}) - V(\vec{r}+\delta\vec{r})\right]$$

$$V(\vec{r}+\delta\vec{r}) = V(\vec{r}) + \nabla V\cdot\delta\vec{r} + \mathcal{O}(\delta\vec{r}^{\,2})$$

$$\begin{aligned}\delta S \equiv S[\vec{r}+\delta\vec{r}] - S[\vec{r}] &= \int_{t_1}^{t_2} dt\,\left[m\dot{\vec{r}}\cdot\delta\dot{\vec{r}} - \nabla V\cdot\delta\vec{r}\right] + \ldots \\ &= \int_{t_1}^{t_2} dt\,\left[-m\ddot{\vec{r}} - \nabla V\right]\cdot\delta\vec{r} + \left[m\dot{\vec{r}}\cdot\delta\vec{r}\right]_{t_1}^{t_2}\end{aligned}$$

Vanishes because we fix the end points

The Proof concludes

$$\delta S = \int_{t_1}^{t_2} dt \; \left[-m\ddot{\vec{r}} - \nabla V \right] \cdot \delta \vec{r}$$

- The condition that the path we started with is an extremum of the action is

$$\delta S = 0$$

- Which should hold for *all* changes $\delta \vec{r}(t)$ that we make to the path. The only way this can happen is if the expression in [...] is zero. This means

$$m\ddot{\vec{r}} = -\nabla V$$

- We recognize this as Newton's equations. Requiring that the action is extremized is equivalent to requiring that the path obeys Newton's equations. $\square$

The Lagrangian

- The integrand of the action is called the *Lagrangian*

$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - V(\vec{r})$$

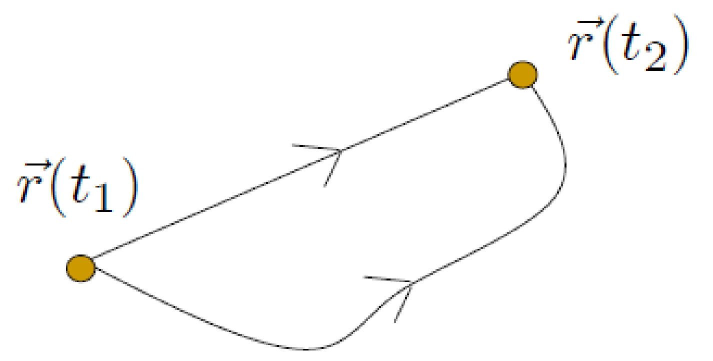
- The *"principle of least action"* is something of a misnomer. The action doesn't have to be minimal. It is often a saddle point.

- This idea is also called *"Hamilton's Principle"*, after Hamilton who gave the general statement some 50 years after Lagrange.

Example 1: free particle motion

$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2}$$

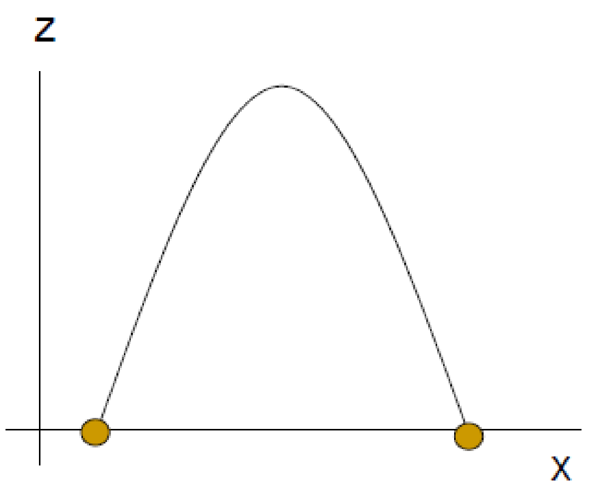
- We want to minimize the kinetic energy over a fixed time.....so the particle must take the most direct route. This is a straight line.
- But do we slow down to begin with, then speed up? Or do we go at a uniform speed?
- To minimize the kinetic energy, we should go at a uniform speed.



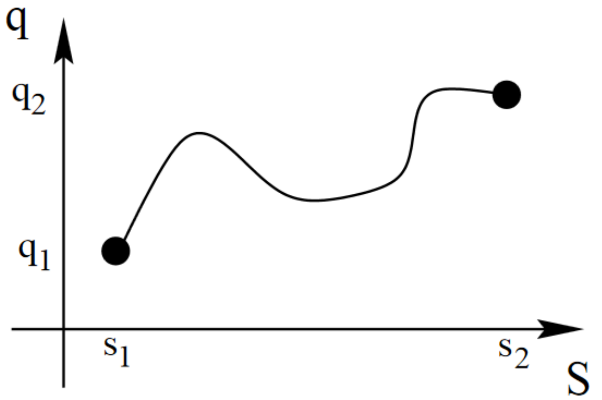
Example 2: Motion in uniform gravity

$$L = \frac{1}{2}m\dot{x}^{\,2} + \frac{1}{2}m\dot{z}^{\,2} - mgz$$

- Now we don't want to go in a straight line. We can minimize the difference between K.E. and P.E. if we go up, where the P.E. is bigger.
- But we don't want to go too high either.
- To strike the right balance, the particle takes a parabola.



Lagrange's equations in general



Imagine we are interested in one curve that minimizes the quantity

$$S[q(s)] = \int_{s_1}^{s_2} L(q, \dot{q}, s) ds$$

$$\delta S = S[\bar{q} + \delta q] - S[\bar{q}]$$

Since $\bar{q}(s)$ is the *minimum* though $\delta S = 0$ to lowest order in δq.

Let's calculate $S[\bar{q} + \delta q]$ to order δq

$$S[\bar{q} + \delta q] \quad = \quad \int_{s_1}^{s_2} L(\bar{q} + \delta q, \dot{\bar{q}} + \delta \dot{q}, s) ds$$

Like our calculation of the special case

Let's calculate $S[\bar{q}+\delta q]$ to order δq

$$\begin{aligned} S[\bar{q}+\delta q] &= \int_{s_1}^{s_2} L(\bar{q}+\delta q, \dot{\bar{q}}+\delta\dot{q}, s)ds \\ &\simeq \int_{s_1}^{s_2} \left(L(\bar{q}, \dot{\bar{q}}, s) + \delta\dot{q}\frac{\partial L}{\partial\dot{q}} + \delta q\frac{\partial L}{\partial q} + ...\right) ds \\ &\simeq S[\bar{q}] + \int_{s_1}^{s_2} \left(\delta\dot{q}\frac{\partial L}{\partial\dot{q}} + \delta q\frac{\partial L}{\partial q}\right) ds + \mathcal{O}(\delta q^2) \end{aligned}$$

Integrating the second term by parts ($u = \partial L/\partial\dot{q}$, $dv/ds = \delta\dot{q}$ etc)

$$\int_{s_1}^{s_2} \delta\dot{q}\frac{\partial L}{\partial\dot{q}} = \left[\delta q\frac{\partial L}{\partial\dot{q}}\right]_{s_1}^{s_2} - \int_{s_1}^{s_2} \delta q\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{q}}\right) ds$$

The first term vanishes since δq vanishes at the ends of the path.

Thus

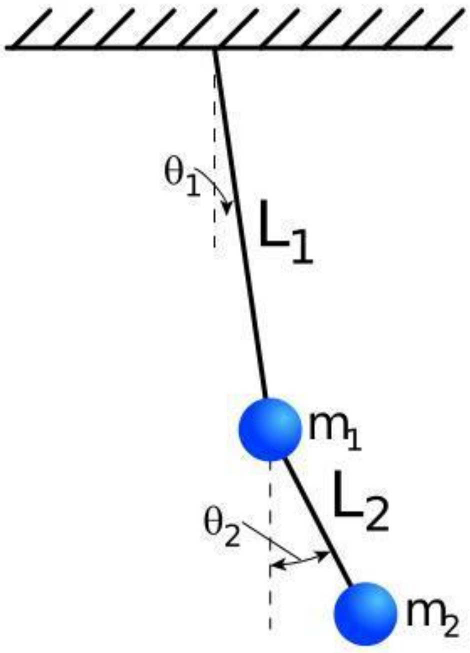
$$S[\bar{q}+\delta q] - S[\bar{q}] = -\int_{s_1}^{s_2} \delta q\left(\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{q}}\right) - \frac{\partial L}{\partial q}\right) ds \; + \; ...$$

This is only zero (at order δq) if

$$\boxed{\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{q}}\right) - \frac{\partial L}{\partial q} = 0}$$

Example 3: Double Pendulum

$$L = \frac{m_1}{2} l_1^2 \dot{\theta}_1^{\,2} + \frac{m_2}{2}(l_1^2 \dot{\theta}_2^{\,2} + l_1^2 \dot{\theta}_1^{\,2} + 2 l_1 l_2 \dot{\theta}_1^{\,2} \dot{\theta}_2^{\,2} cos(\theta_1 - \theta_2)) + m_1 g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2 + m_2 g l_1 cos\theta_1$$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

Makes analysis much easier!

Noether's Theorem (1915)

Invariance of the Lagrangian ↔ conservation laws

No explicit dependence of L on t
Time translation invariant ↔ energy conserved

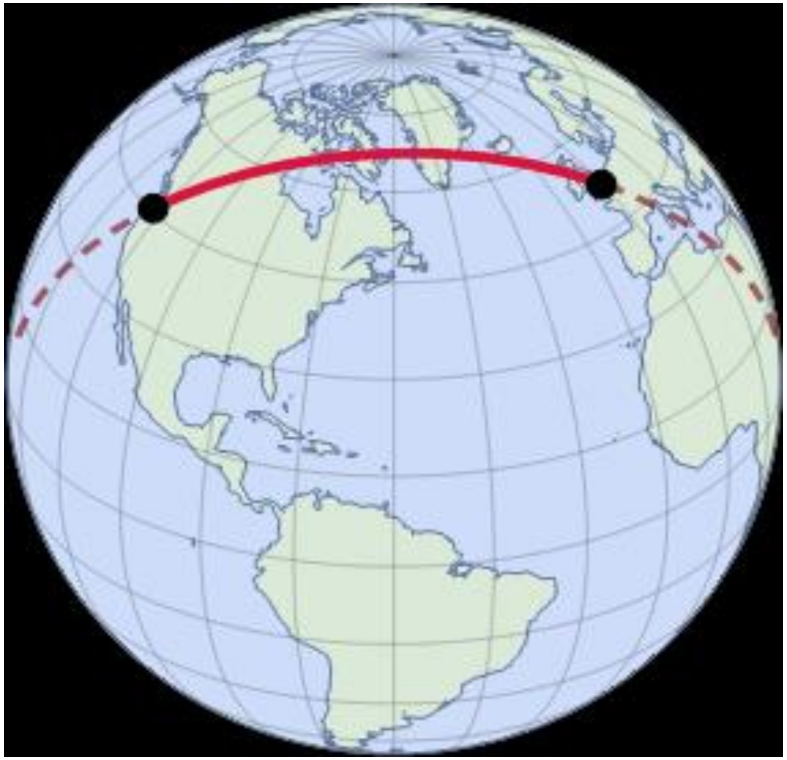
No dependence of L on direction θ ↔ angular momentum conserved

L is position invariant ↔ momentum is conserved

No preferred time, place or direction means
Conservation of energy, momentum and angular momentum

If a system has a continuous symmetry then there must be conserved quantities

Paths on Curved Surfaces



$$
\begin{aligned}
ds^2 &= g_{ab}dx^a dx^b \\
0 &= \frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} \\
\Gamma^a_{bc} &= \frac{1}{2}g^{an}\left[\frac{\partial g_{bn}}{\partial x^c} + \frac{\partial g_{cn}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^n}\right]
\end{aligned}
$$

Crucial for finding equations of motion in general relativity where space and time are curved by the presence of mass-energy

Why use an Action approach?

- There are several reasons to use this approach.
  - It is independent of the coordinates we choose to work in. The idea of minimizing the action holds in Cartesian coordinates, polar coordinates, rotating frames, or any other system of coordinates you choose to work in. This can often be very useful.
  - It is easy to implement constraints in this set-up
- This means that we can solve rather tricky problems, such as the strange motion of spinning tops, with ease.
  - All of this will be covered in the third year "Classical Dynamics" course.

An Action to unify all of physics?!

- All fundamental laws of physics can be expressed in terms of a least action principle. This is true for electromagnetism, special and general relativity, particle physics, and even more speculative pursuits that go beyond known laws of physics such as string theory.

- For example, (nearly) every experiment ever performed can be explained by the Lagrangian of the standard model

$$\mathcal{L} = \sqrt{g}(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma_\mu D^\mu \psi + |\mathcal{D}h|^2 - V(|h|) + \psi\psi h)$$

Einstein

Maxwell
Yang-Mills

Dirac

Higgs

Yukawa

Seeking out the paths?

- The principle of least action gives a very different way of looking at things:

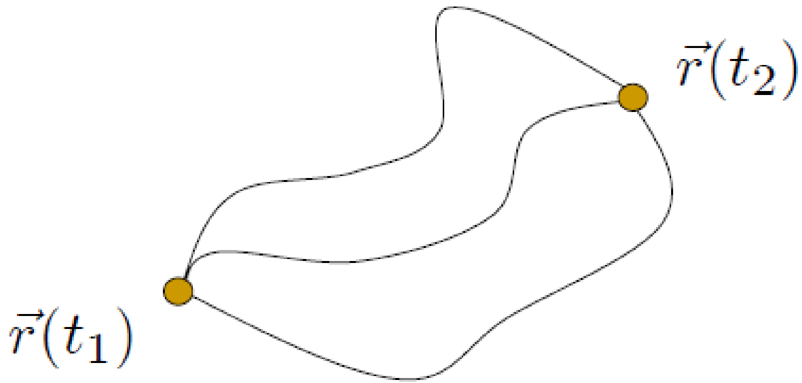
- In the Newtonian framework, you start to develop an intuition for how particles move, which goes something like this: at each moment in time, the particle thinks “where do I go now?”. It looks around, sees the potential, differentiates it and says “ah-ha….I go this way”. Then, an infinitesimal moment later, it does it all again.

- But the Lagrangian framework suggests a rather different viewpoint. Now the particle is taking the path which is minimizing the action. How does it know this is the minimal path? Is it sniffing around, checking out all paths, before it decides: “I think I’ll go this way”.

- On some level, this philosophical pondering is meaningless. After all, we just proved that the two ways of doing things are completely equivalent. However, the astonishing answer is: yes, the particle does sniff out every possible path! This is the way *quantum mechanics* works.

Feynman's Path Integral

- Nature is probabilistic. At the deepest level, things happen by random chance. This is the key insight of quantum mechanics.



- The probability that a particle starting at $\vec{r}(t_1)$ will end up at $\vec{r}(t_2)$ is expressed in terms of an *Amplitude* A, which is a complex number that can be thought of as the square root of the probability

$$Prob = |A|^2$$

Evaluating the Path Integral

- To compute the amplitude, you must sum over all paths that the particle takes, weighted with by phase

$$A = \sum_{\text{paths}} \exp(iS/\hbar)$$

- Here S is the action, while $\hbar$ is Planck's constant (divided by 2π). It's a fundamental constant of Nature.

- The way to think about this is that when a particle moves, it really does take all possible paths. Away from the classical path, the action varies wildly, and the sum of different phases averages to zero. Only near the classical path do the phases reinforce each other.

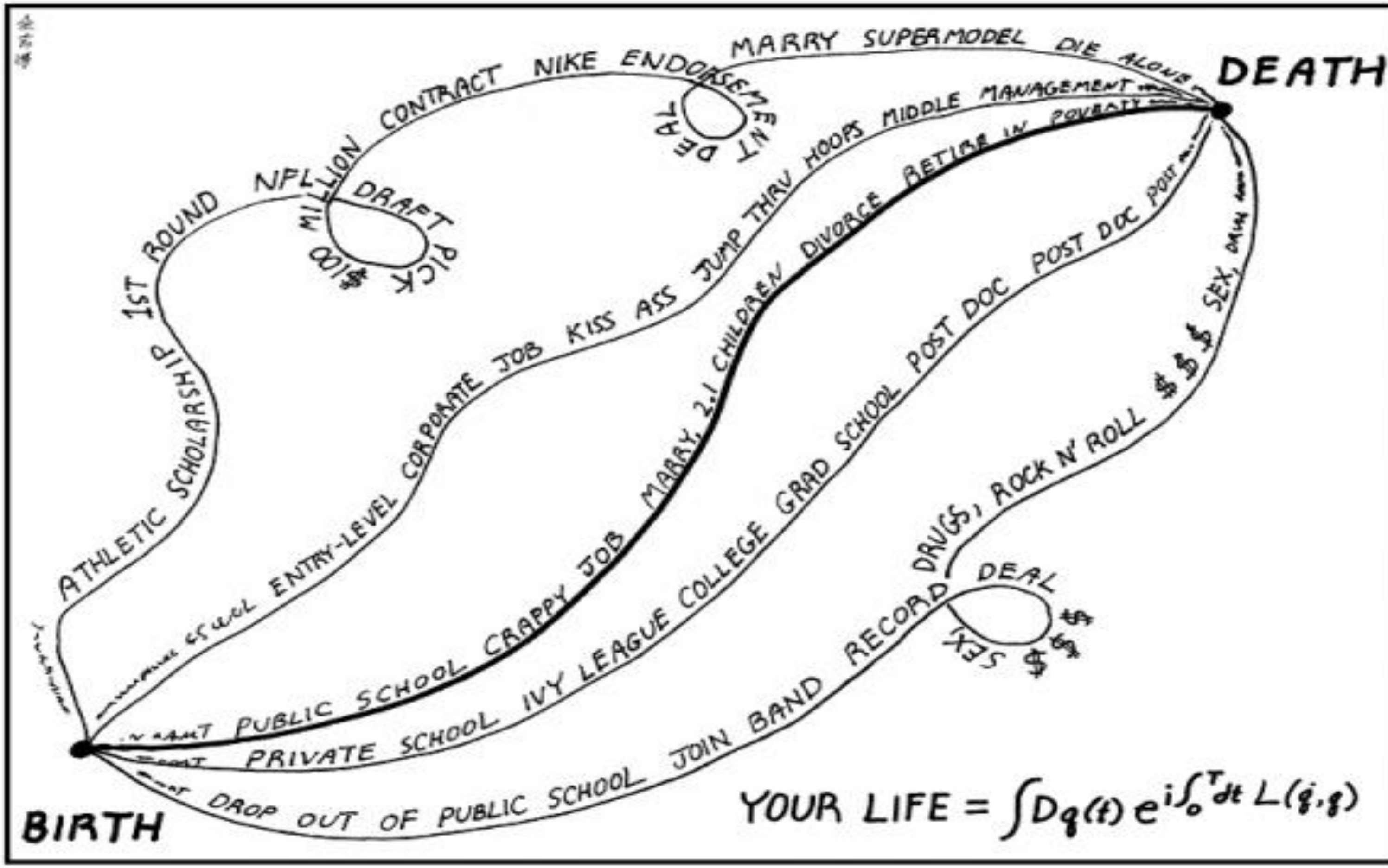
- You will learn more about this in various courses on quantum mechanics over the next few years.

Feynman’s sum over histories

- ‘Thirty-one years ago [in 1949], Dick Feynman told me about his "sum over histories" version of quantum mechanics. "The electron does anything it likes," he said. "It just goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave function." I said to him, "You're crazy." But he wasn't.’

  --Freeman J. Dyson, 1980.

A path-integral formulation of your life!



Duncan O'Dell

Further reading

- R. Feynman, *Feynman Lectures in Physics,* Addison-Wesley 1970, chapter 19, (free online at http://www.feynmanlectures.caltech.edu/II_19.html)
- L. Susskind and G. Hrabovsky, *Classical Mechanics*, Penguin, 2014
- C. Lanczos, *The Variational Principles of Mechanics*, Dover (reprint of 1970 edn)