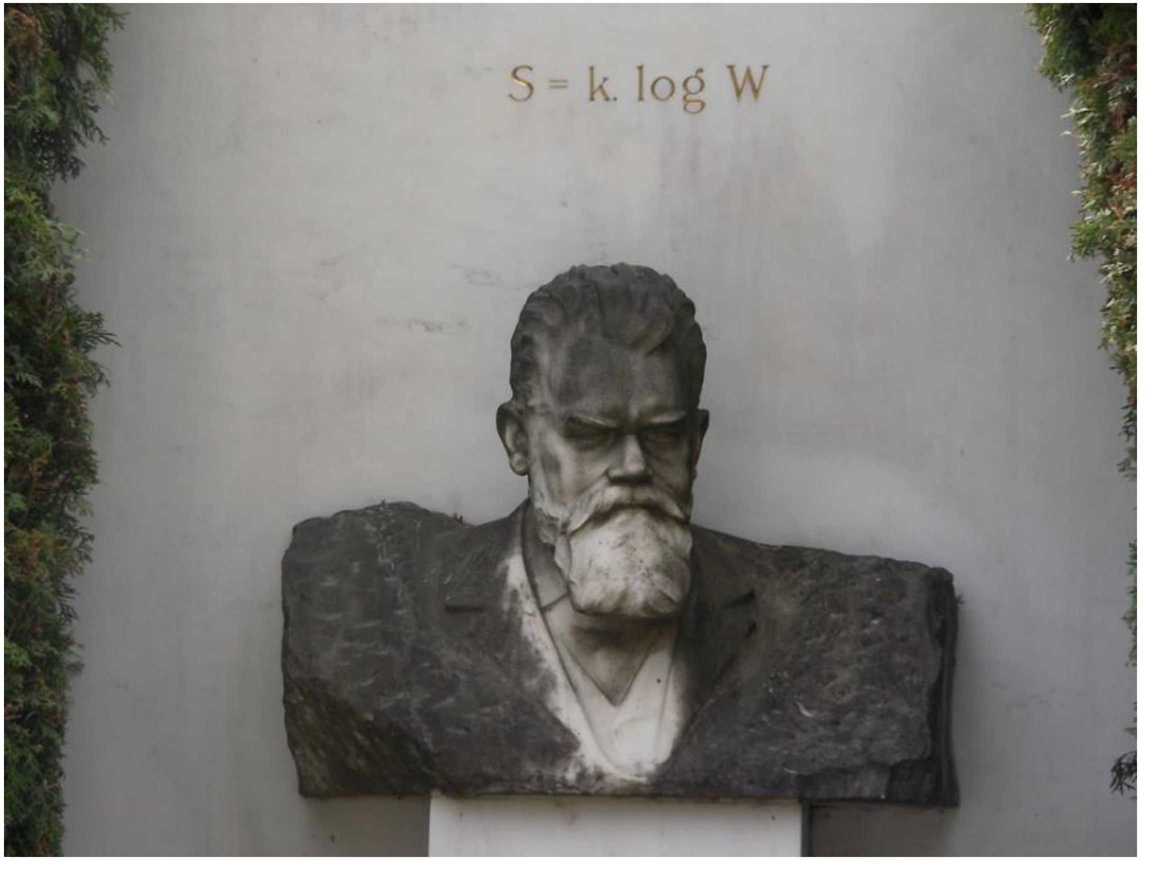
Concepts in Theoretical Physics

Lecture 5: Statistical Physics: Entropy, Demons and Black Holes

John D Barrow

'The laws of thermodynamics:
You can't win
You can't break even
You can't get out of the game'
Charlie Smalls

A famous formula



Ludwig Boltzmann (1844-1906)

Things always tend to get worse

The final mixed state is less ordered, or high entropy.

The Second Law of Thermodynamics:
Entropy ('disorder') of an isolated system never decreases

The Reversibility Paradox

We see evolution from Order → Disorder

Yet the microscopic equations of motion

$$d^2\underline{x}/dt^2 = F(\underline{x})$$

are invariant under $t \rightarrow -t$

Why don't we see the time-reversed process ???

How can you solve 10^{24} equations at once?

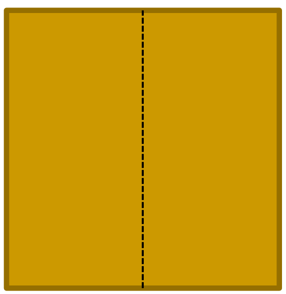
Use a statistical approach with three new concepts:

Microstates, Macrostates and Statistical Entropy

Put N particles in a box and track the number N_L in the left-hand half (state L) and the number N_R in the right-hand half (state R)

The microstate is the list all the 2^N states that a system of N particles could occupy.

For example, if N = 10, the list of these would look like



LLLLLLLLLL

RLLLLLLLLL

LRLLRRLLLL

etc….

RRRRRRRRRR

Collect them all into one number by defining

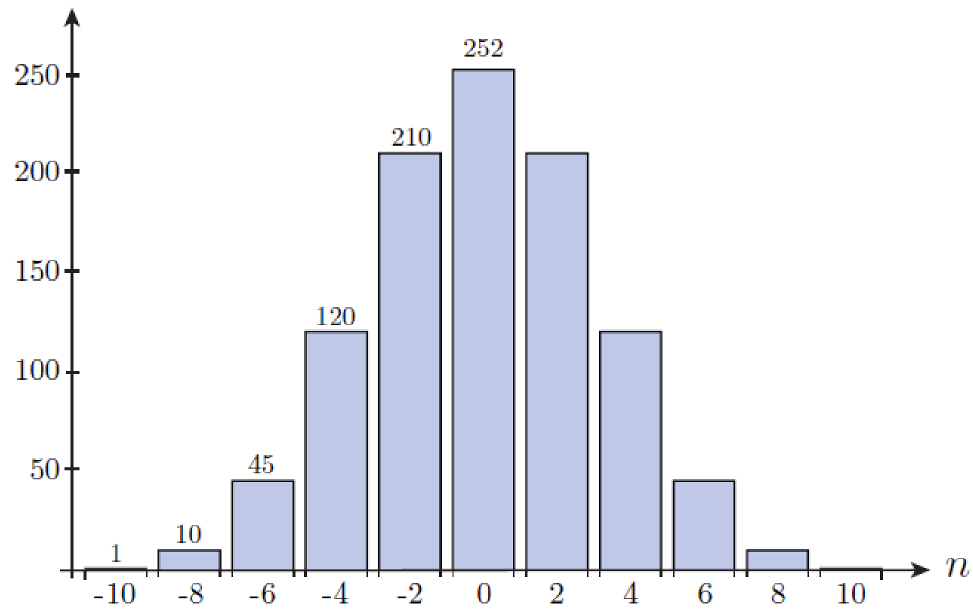
$$n = N_L - N_R$$

This labels the macrostate of the system

Eg. when N = 10 we can have n = +10 or -10, with 1 microstate each

Or n = +8 or -8 with 10 possible microstates (ways of having one R) etc

The statistical spread for N =10



It is easy to generalize this: let $W(n)$ be the number of ways to have N particles with N_L particles on the left and N_R particles on the right. The answer is

$$W(n) = \frac{N!}{N_L! N_R!} = \frac{N!}{(\frac{N-n}{2})!(\frac{N+n}{2})!} . \qquad (3.3.2)$$

For large n, approximately: $W(n) \approx 2^N \exp[-n^2/2N]$

Defining statistical entropy

The fundamental assumption of statistical mechanics

Each accessible microstate is equally likely

The entropy of a macrostate is given by

the logarithm of the number of microstates:

$$S = k \log W$$

$k = 1.38 \times 10^{-23}\ JK^{-1}$ is Boltzmann's constant

(it merely converts temperature to energy units)

We expect evolution to the macrostate with the largest number of

(equally likely) microstates and therefore that

$$dS/dt \geq 0$$

Combine 2 systems W_1 and W_2 and the number of microstates multiply: $W = W_1 W_2$

But the entropy is additive: $S = k\log W = k\log W_1 + k\log W_2 = S_1 + S_2$

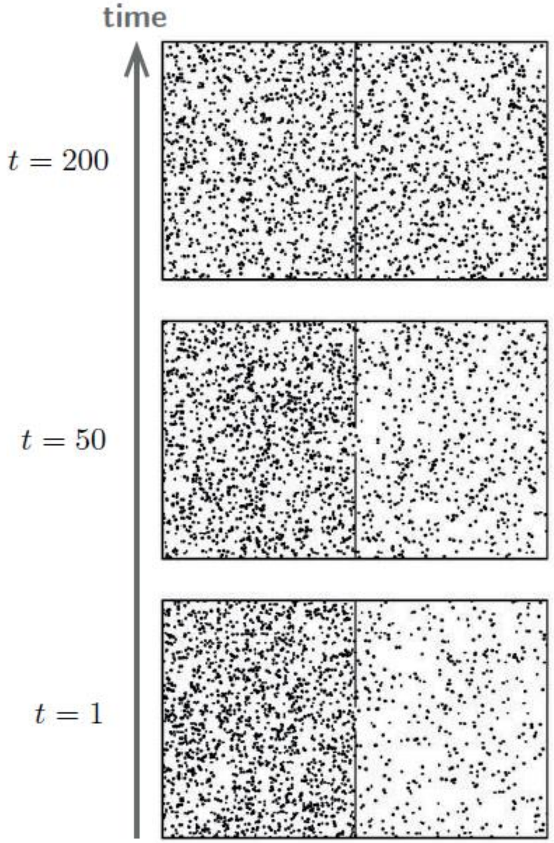
The laws of thermodynamics

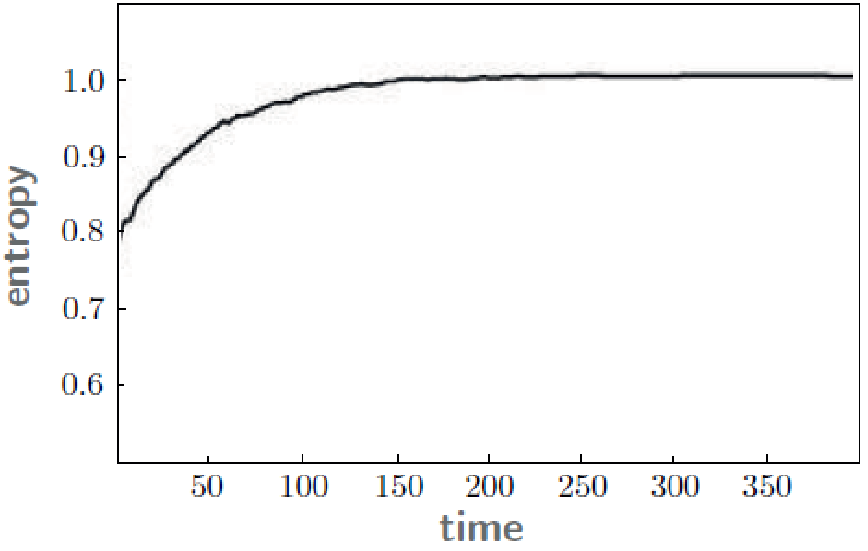
- Zeroth law: If two systems are both in thermal equilibrium with a third system then they are in thermal equilibrium with each other (T = constant).
- 1st Law: Energy is conserved $dE = TdS + PdV$ ← external work
- 2nd Law: Entropy is non-decreasing $dS/dt \geq 0$
- 3rd Law: temperature cannot be reduced to absolute zero in a finite number of steps

Homer Simpson: 'In this house we obey the laws of thermodynamics'

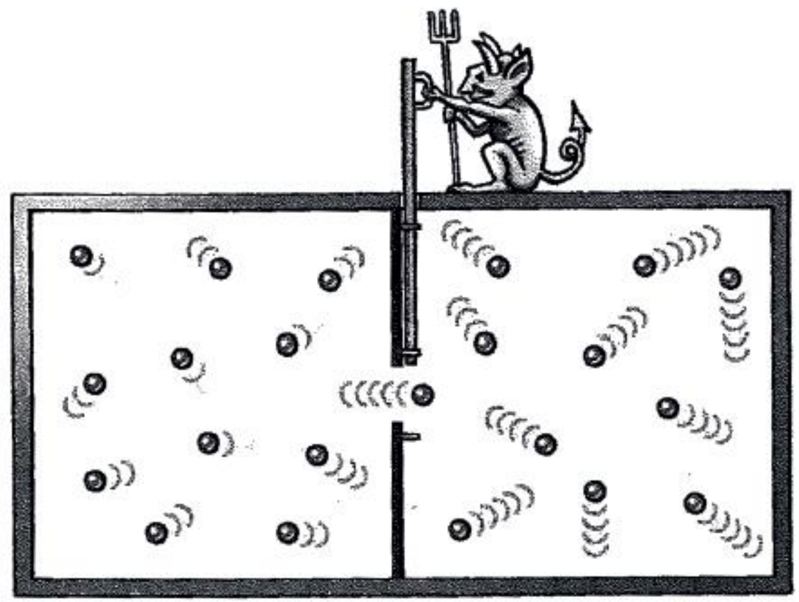
The Arrow of Time

Box of 2000 molecules with 1600 on the Left and 400 on the Right at t = 1.
There is gradual mixing on a macro level even though micro collisions are time symmetric
Plot the entropy versus time and we see it increasing towards the maximum ('equilibrium')





Maxwell's Sorting Demon (1871)

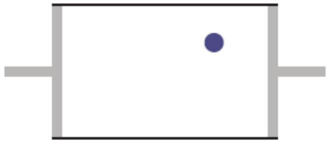


The first entry of 'mind' into physics challenged the 2nd Law of thermodynamics.
A Demon only allows faster than average (= hot) molecules through his gate
The right-hand side gets hot and the left-hand end gets cool, as time passes
Anti-mixing and $dS/dt < 0$
You could use the temperature gradient to drive a perpetual motion machine!

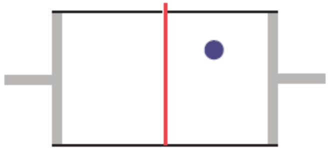
Entropy and Information

Leo Szilard (1929) taught us: Information is Physical – it allows us to do work

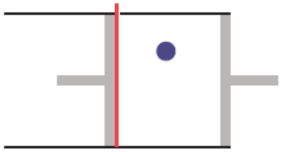
Possessing information allows us to extract useful work from a system in ways that would have otherwise been impossible. Szilard arrived at these insights through a clever new version of Maxwell's demon: this time there is only a single molecule in the box. Two walls of the box are replaced by movable pistons



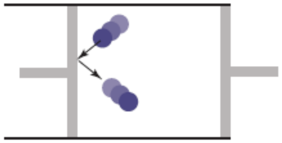
A partition (now without a hole) is placed in the middle. The molecule is on one side and the other side is empty



The demon measures and records on what side of the partition the gas molecule is, gaining one *bit* of information. He then pushes in the piston that closed off the empty half of the box



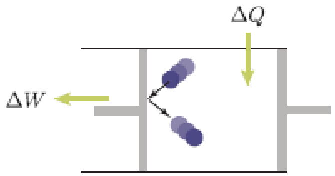
In the absence of friction, this process doesn't require any energy. Note the crucial role played by information in this setup. If the demon didn't know which half of the box the molecule was in, he wouldn't know which piston to push in. After removing the partition, the molecule will push against the piston and the one-molecule gas "expands"



Entropy and Information continued

ΔW = work done

In this way we can use the system to do useful work (e.g. by driving an engine). Where did the energy come from? From the heat Q of the surroundings (with temperature T),



The work done when the gas expands from $V_i = V$ to $V_f = 2V$ is given by a standard formula in thermodynamics:

$$\Delta W = kT \log\left(\frac{V_f}{V_i}\right) = kT \log 2\ . \qquad (3.4.7)$$

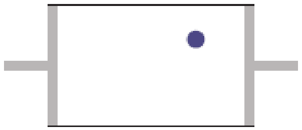
Recall that $\mathrm{d}W = F\mathrm{d}x = p\mathrm{d}V$, where p is the pressure of the gas. The integrated work done is therefore

$$\Delta W = \int_{V_i}^{V_f} p\,\mathrm{d}V\ .$$

Using the ideal gas law for the one-molecule gas, $pV = kT$, we can write this as

$$\Delta W = \int_{V_i}^{V_f} \frac{kT}{V}\,\mathrm{d}V = kT \log\left(\frac{V_f}{V_i}\right)\ .$$

The system returns back to its initial state



This completes one cycle of operation. The whole process is repeatable. Each cycle would allow extraction and conversion of heat from the surroundings into useful work in a cyclic process. The demon seems to have created a perpetual motion machine of the second kind.[7] In particular, in each stage of the cycle the entropy decreases by $\Delta S = \Delta Q/T$ (another classic formula of thermodynamics). Using $\Delta Q = -\Delta W$, we find

$$\Delta S = -k \log 2\ . \qquad (3.4.8)$$

Szilard's demon again seems to have violated the Second Law.

Rescuing the Second Law

In 1982, Charles Bennett observed that Szilard's engine is not quite a closed cycle.[8] While after each cycle the box has returned to its initial state, the mind of the demon has not! He has gained one bit of recorded information. The demon needs to erase the information stored in his mind in order for the process to be truly cyclic. However, Rolf Landauer had shown[9] in 1961 that the erasure of information is necessarily an irreversible process.[10] In particular, destroying one bit of information increases the entropy of the world by at least

$$\Delta S \geq k \log 2 \ . \tag{3.4.9}$$

So here is the modern resolution of Maxwell's demon: the demon must collect and store information about the molecule. If the demon has a finite memory capacity, he cannot continue to cool the gas indefinitely; eventually, information must be erased. At that point, he finally pays the entropy bill for the cooling he achieved. (If the demon does not erase his record, or if we want to do the thermodynamic accounting before the erasure, then we should associate some entropy with the recorded information.)

8 See https://arxiv.org/abs/physics/0210005

How can you solve 10^{11} equations?

How does a galaxy work?

$$\begin{aligned} \ddot{x}_i &= -\nabla\Phi(x_{1,\ldots,}x_n) \quad i = 1,..,n \\ \nabla^2\Phi &= 4\pi G\rho \end{aligned}$$

But n = 10^{11} !!

How to deal with a 100 billion stars

Define a distribution function $f(x, v, t)$

$fd^3xd^3v =$ mass of stars in volume $(x, x+\delta^3 x)$ with velocity in $(v, v+\delta^3 v)$ at time t

Therefore integrating over the velocities

$$\rho = \int f d^3 v \qquad (1)$$

and Poisson's equation gives

$$\nabla^2 \Phi = 4\pi G \rho = \int f d^3 v \qquad (2)$$

If no stars are created or destroyed or suffer close encounters then $f(x, v, t)$ is conserved and satisfies Boltzmann's equation:

$$0 = \frac{df(x,v,t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt}$$

[If there are collisions, creations, or encounters then the left-hand side will be non-zero.]

But from Newton's laws:

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = -\frac{\partial \Phi}{\partial x}$$

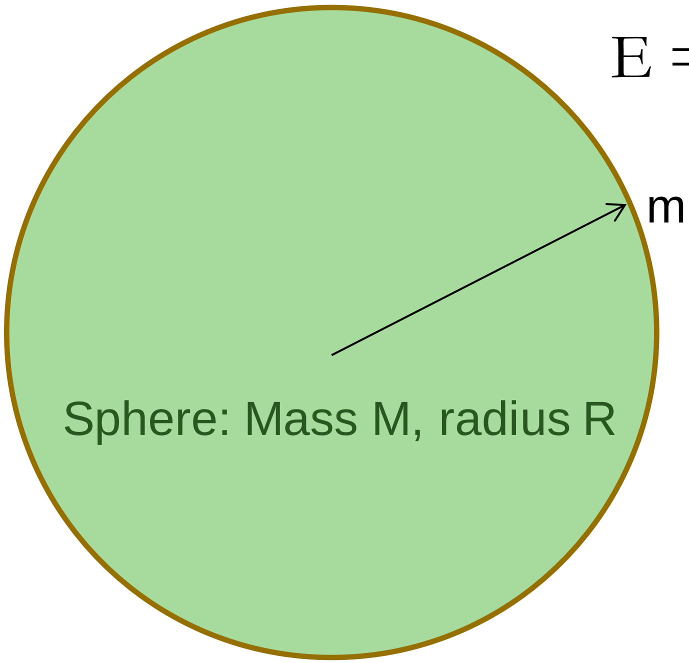
Therefore, in a steady state ($\frac{\partial f}{\partial t} = 0$) we have

$$0 = \frac{\partial f}{\partial x} \cdot v - \frac{\partial f}{\partial v} \cdot \frac{\partial \Phi}{\partial x} \qquad (3)$$

We need to solve eqns. (1-3), for f, Φ and ρ simultaneously. We can use symmetry to choose a form for f (eg spherically or cylindrically symmetric disk) and solve for Φ an then ρ. Or assume a form for Φ, or ρ, and solve for f.

Boundary conditions:
$f \rightarrow 0$ as $|v|$ and $|x| \rightarrow \infty$

How fast to escape from gravity?



$E = \frac{1}{2} mV^2 - GMm/R \geq 0$ to escape to ∞

$$V \geq V_{esc} = \sqrt{2GM/R}$$

$V_{esc}(\text{Earth}) = 11.2$ km/s

$V(\text{rifle bullet}) \approx 1.7$km/s

$V_{esc} = c$ for $R = R_s = 2GM/c^2$ is a surface of no escape in general relativity (<u>not just to escape to infinity</u>)

Black Holes Need Not Be Extreme

Black Hole Density = Mass / volume $\propto M/R_s^3 \propto M/M^3 \propto 1/M^2$

Density is less than air for $M > 10^9\ M_{sun}$

Small black holes are more extreme

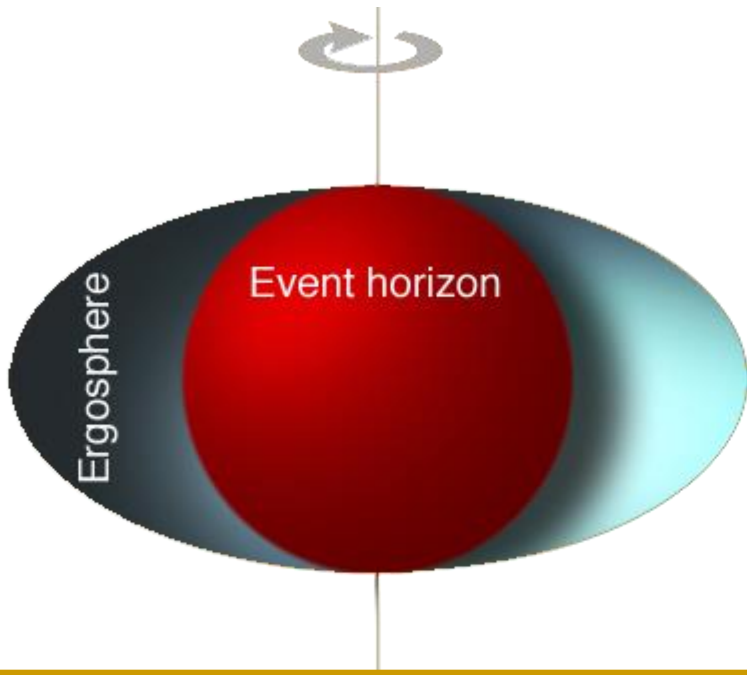
If you fall in then density $\rightarrow \infty$ as you fall to the centre

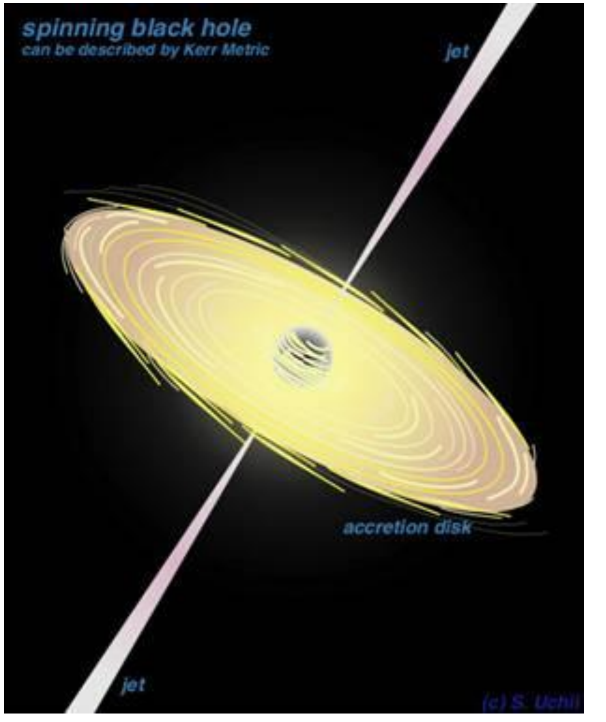
What is a black hole?

M87 Event Horizon Telescope

- Escape velocity $V = (2GM/R)^{1/2} = c$
- $R = 2GM/c^2$
- Not solid objects -- density $\propto M/R^3 \propto 1/M^2$
- Simplest objects in the universe
- Only external observable properties are :
  mass, charge, and angular momentum
- Things go in. Nothing comes out.
- Form when massive stars die (if $M > 3M_{sun}$) and power quasars and galactic nuclei when $M \approx (10^6\text{-}10^9)M_{sun}$
- Can form in the early universe with all masses (even $M << 3M_{sun}$)

Kerr Rotating Black Hole (1963)





$$R = \frac{1}{2}\{R_s + \sqrt{(R_s^2 - 4a^2)}\}$$

where $R_s \equiv 2GM/c^2$ and $a \equiv J/Mc$ so $J \leq GM^2/c$

J is the angular momentum of the black hole, M is its mass

Black hole mechanics

Laws of Black Hole Mechanics

Zeroth: $g = GM/R^2$ constant on horizon

First: $\delta(Mc^2) = \delta A\, c^6/32\pi G^2 + P\delta V \leftarrow$ external work

Second: $\delta A \geq 0$

Third: You can't reduce g to zero in a finite number of steps

Black hole thermodynamics

Laws of Black Hole Mechanics	Laws of Thermodynamics
Zeroth: $g = GM/R^2$ constant on horizon	T = constant
First: $\delta(Mc^2) = \delta A\, c^6/32\pi G^2 + P\delta V$	$\delta E = T\,\delta S + P\delta V$
Second: $\delta A \geq 0$	$\delta S \geq 0$
Third: You can't reduce g to zero in a finite number of steps	You can't reduce T to absolute zero in a finite number of steps

IDENTICAL IF AND ONLY IF

Black hole entropy $S = kc^3 A/4G\hbar$ and temperature $T = \hbar g/2\pi ck = \hbar c^3/8\pi kGM$

Links relativity (c), quantum (h), thermal (k) and gravitational (G) physics

$$S = kc^3 A/4G\hbar = (kc^3/4G\hbar) \times 4\pi(2GM/c^2)^2$$

$$S = 4\pi k\, GM^2/\hbar c = 4\pi k(M/m_{pl})^2, \quad m_{pl} = 10^{-5}\ \text{gm}$$

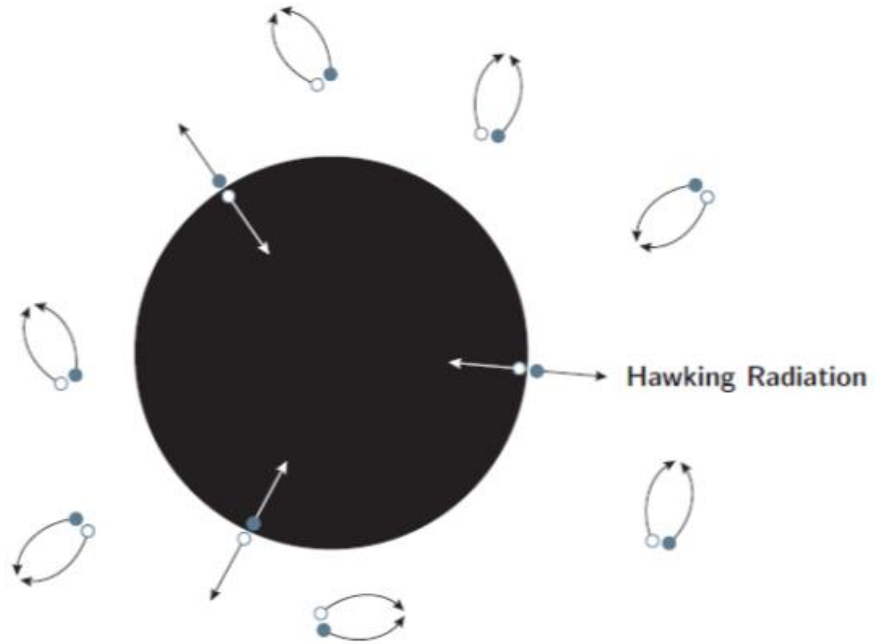
$$T = \hbar c^3/8\pi kGM \rightarrow \infty \text{ as } M \rightarrow 0$$

Black hole evaporation and explosions
Hawking, 1975

$dE/dt = d(Mc^2)/dt = \text{area} \times aT^4 \times c \propto M^2 \times M^{-4}$ so Lifetime $= \int dt \propto \int M^2 dM \propto M^3$

Lifetime = 10 billion yrs $\times$ $(M/10^{14}\ \text{gm})^3$

Quantum black holes are black bodies

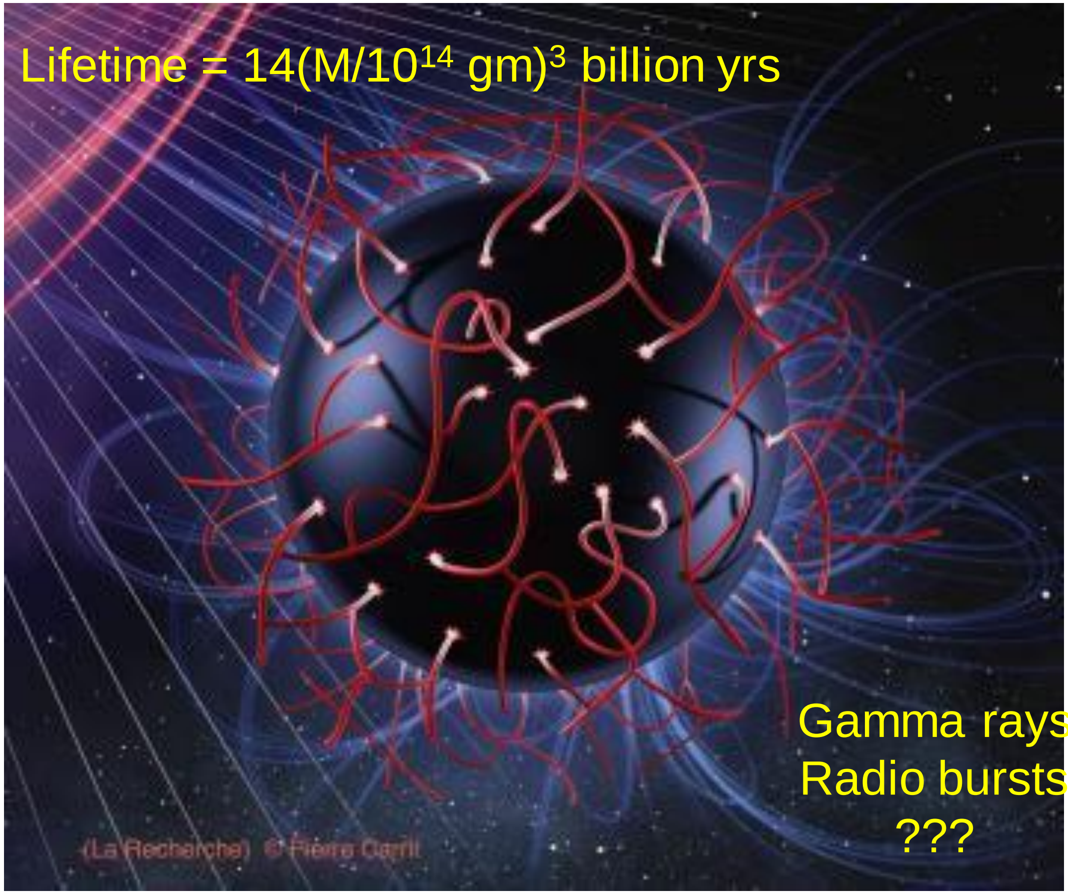


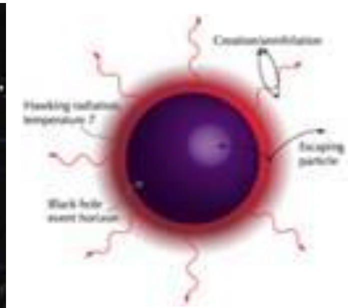
$\Delta E \Delta t \sim \hbar/2$ and $\Delta E \sim E \sim kT_{bh}$

$\Delta t \sim R_s/c \sim 2GM/c^3$

$T_{bh} \sim \hbar c^3/kGM$

Evaporation of Black Holes (1974)





Where does all the information go?

Further reading

- J. Gleick *The Information: a natural history of information theory*, Pantheon Books
- H.S. Leff and A.F. Rex, (editors), *Maxwell's Demon: Entropy, information, computing*. This volume contains reprints of all the key articles, including those by Maxwell, Szilard, Landauer and Bennett, together with a very nice overview of the history and key insights.
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