‘The laws of thermodynamics:
You can’t win
You can’t break even
You can’t get out of the game’
Charlie Smalls
A famous formula

Ludwig Boltzmann (1844-1906)
Things always tend to get worse

The Second Law of Thermodynamics:
Entropy ("disorder") of an isolated system never decreases

The final mixed state is less ordered, or high entropy.
The Reversibility Paradox

We see evolution from Order $\rightarrow$ Disorder
Yet the microscopic equations of motion
\[ \frac{d^2x}{dt^2} = F(x) \]
are invariant under $t \rightarrow -t$
Why don’t we see the time-reversed process ??
How can you solve \(10^{24}\) equations at once?

Use a statistical approach with three new concepts: **Microstates, Macrostates and Statistical Entropy**

Put \(N\) particles in a box and track the number \(N_L\) in the left-hand half (state L) and the number \(N_R\) in the right-hand half (state R)

The microstate is the list all the \(2^N\) states that a system of \(N\) particles could occupy.

For example, if \(N = 10\), the list of these would look like

- \(LLLLLLLLLL\)
- \(RLLLLLLL\)
- \(LRLLRRLLL\)
- etc....
- \(RRRRRRRRRR\)

Collect them all into one number by defining

\[ n = N_L - N_R \]

This labels the macrostate of the system

Eg. when \(N = 10\) we can have \(n = +10\) or \(-10\), with 1 microstate each

Or \(n = +8\) or \(-8\) with 10 possible microstates (ways of having one R) etc
The statistical spread for $N = 10$

It is easy to generalize this: let $W(n)$ be the number of ways to have $N$ particles with $N_L$ particles on the left and $N_R$ particles on the right. The answer is

$$W(n) = \frac{N!}{N_L!N_R!} = \frac{N!}{(N-n)!\left(\frac{N+n}{2}\right)!}.$$ \hspace{1cm} (3.3.2)

For large $n$, approximately: $W(n) \approx 2^N \exp[-n^2/2N]$
Defining statistical entropy

The fundamental assumption of statistical mechanics

Each accessible microstate is equally likely

The entropy of a macrostate is given by
the logarithm of the number of microstates:

\[ S = k \log W \]

\( k = 1.38 \times 10^{-23} \text{ JK}^{-1} \) is Boltzmann’s constant
(it merely converts temperature to energy units)

We expect evolution to the macrostate with the largest number
of
(equally likely) microstates and therefore that

\[ \frac{dS}{dt} \geq 0 \]

Combine 2 systems \( W_1 \) and \( W_2 \) and the number of microstates multiply: \( W = W_1 W_2 \)
But the entropy is additive: \( S = k \log W = k \log W_1 + k \log W_2 = S_1 + S_2 \)
The laws of thermodynamics

- **Zeroth law**: If two systems are both in thermal equilibrium with a third system then they are in thermal equilibrium with each other \((T = \text{constant})\).

- **1st Law**: Energy is conserved \(dE = TdS + dW\)

- **2nd Law**: Entropy is non-decreasing \(\frac{dS}{dt} \geq 0\)

- **3rd Law**: Temperature cannot be reduced to absolute zero in a finite number of steps

Homer Simpson: ‘In this house we obey the laws of thermodynamics’
The Arrow of Time

Box of 2000 molecules with 1600 on the Left and 400 on the Right at \( t = 1 \). There is gradual mixing on a macro level even though micro collisions are time symmetric.

Plot the entropy versus time and we see it increasing towards the maximum (‘equilibrium’).
Maxwell’s Sorting Demon (1871)

The first entry of ‘mind’ into physics challenged the 2nd Law of thermodynamics.

A Demon only allows faster than average (= hot) molecules through his gate.

The right-hand side gets hot and the left-hand end gets cool, as time passes.

Anti-mixing and $\frac{dS}{dt} < 0$

You could use the temperature gradient to drive a perpetual motion machine!
Entropy and Information

Leo Szilard (1929) taught us: Information is Physical – it allows us to do work

Possessing information allows us to extract useful work from a system in ways that would have otherwise been impossible. Szilard arrived at these insights through a clever new version of Maxwell’s demon: this time there is only a single molecule in the box. Two walls of the box are replaced by movable pistons.

A partition (now without a hole) is placed in the middle. The molecule is on one side and the other side is empty.

The demon measures and records on what side of the partition the gas molecule is, gaining one bit of information. He then pushes in the piston that closed off the empty half of the box.

In the absence of friction, this process doesn’t require any energy. Note the crucial role played by information in this setup. If the demon didn’t know which half of the box the molecule was in, he wouldn’t know which piston to push in. After removing the partition, the molecule will push against the piston and the one-molecule gas “expands.”
In this way we can use the system to do useful work (e.g. by driving an engine). Where did the energy come from? From the heat $Q$ of the surroundings (with temperature $T$).

The work done when the gas expands from $V_i = V$ to $V_f = 2V$ is given by a standard formula in thermodynamics:

$$\Delta W = kT \log \left( \frac{V_f}{V_i} \right) = kT \log 2.$$  \hspace{1cm} (3.4.7)

Recall that $dW = Fdx = p \, dV$, where $p$ is the pressure of the gas. The integrated work done is therefore

$$\Delta W = \int_{V_i}^{V_f} p \, dV.$$

Using the ideal gas law for the one-molecule gas, $pV = kT$, we can write this as

$$\Delta W = \int_{V_i}^{V_f} \frac{kT}{V} \, dV = kT \log \left( \frac{V_f}{V_i} \right).$$

The system returns back to its initial state.

This completes one cycle of operation. The whole process is repeatable. Each cycle would allow extraction and conversion of heat from the surroundings into useful work in a cyclic process. The demon seems to have created a perpetual motion machine of the second kind. In particular, in each stage of the cycle the entropy decreases by $\Delta S = \Delta Q / T$ (another classic formula of thermodynamics). Using $\Delta Q = -\Delta W$, we find

$$\Delta S = -k \log 2.$$  \hspace{1cm} (3.4.8)

Szilard’s demon again seems to have violated the Second Law.
Rescuing the Second Law

In 1982, Charles Bennett observed that Szilard’s engine is not quite a closed cycle. While after each cycle the box has returned to its initial state, the mind of the demon has not! He has gained one bit of recorded information. The demon needs to erase the information stored in his mind in order for the process to be truly cyclic. However, Rolf Landauer had shown in 1961 that the erasure of information is necessarily an irreversible process. In particular, destroying one bit of information increases the entropy of the world by at least

$$\Delta S \geq k \log 2.$$  \hspace{1cm} (3.4.9)

So here is the modern resolution of Maxwell’s demon: the demon must collect and store information about the molecule. If the demon has a finite memory capacity, he cannot continue to cool the gas indefinitely; eventually, information must be erased. At that point, he finally pays the entropy bill for the cooling he achieved. (If the demon does not erase his record, or if we want to do the thermodynamic accounting before the erasure, then we should associate some entropy with the recorded information.)

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9 Landauer, “Irreversibility and Heat Generation in the Computing Process”.
10 In other words, you can’t erase information if you are part of a closed system operating under reversible laws. If you were able to erase information entirely, how would you ever be able to reverse the evolution of the system? If erasure is possible, either the fundamental laws are irreversible—in which case it is not surprising that you can lower the entropy—or you’re not really in a closed system. The act of erasing information necessarily transfers entropy to the outside world.
How can you solve $10^{11}$ equations?

How does a galaxy work?

\begin{align*}
\ddot{x}_i &= -\nabla \Phi(x_1, \ldots, x_n) \quad i = 1, \ldots, n \\
\nabla^2 \Phi &= 4\pi G \rho
\end{align*}

But $n = 10^{11}$!!
How to deal with a 100 billion stars

Define a distribution function \( f(x, v, t) \)

\[
fd^3x d^3v = \text{mass of stars in volume } (x, x+\delta x) \text{ with velocity in } (v, v+\delta v) \text{ at time } t
\]

Therefore integrating over the velocities

\[
\rho = \int fd^3v \tag{1}
\]

and Poisson’s equation gives

\[
\nabla^2 \Phi = 4\pi G \rho = \int fd^3v \tag{2}
\]

If no stars are created or destroyed or suffer close encounters then \( f(x, v, t) \) is conserved and satisfies Boltzmann’s equation:

\[
0 = \frac{df(x, v, t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt}
\]

[If there are collisions, creations, or encounters then the left-hand side will be non-zero.]

But from Newton’s laws:

\[
\frac{dx}{dt} = v \text{ and } \frac{dv}{dt} = -\frac{\partial \Phi}{\partial x}
\]

Therefore, in a steady state \( (\frac{df}{dt} = 0) \) we have

\[
0 = \frac{\partial f}{\partial x} \cdot v - \frac{\partial f}{\partial v} \cdot \frac{\partial \Phi}{\partial x} \tag{3}
\]

We need to solve eqns. (1-3), for \( f, \Phi \) and \( \rho \) simultaneously. We can use symmetry to choose a form for \( f \) (e.g., spherically or cylindrically symmetric disk) and solve for \( \Phi \) and then \( \rho \). Or assume a form for \( \Phi \), or \( \rho \), and solve for \( f \).
How fast to escape from gravity?

$E = \frac{1}{2} mV^2 - \frac{GMm}{R} \geq 0 \text{ to escape to } \infty$

$V_{\text{esc}} = \sqrt{2GM/R}$

$V_{\text{esc}}(\text{Earth}) = 11.2 \text{ km/s}$

$V(\text{rifle bullet}) \approx 1.7\text{km/s}$

$V_{\text{esc}} = c$ for $R = R_s = 2GM/c^2$ is a surface of no escape in general relativity (not just no escape to infinity)
Black Holes Need Not Be Extreme

Black Hole Density = Mass / volume \( \propto \frac{M}{R_s^3} \propto \frac{M}{M^3} \propto \frac{1}{M^2} \)

Density is less than air for \( M > 10^9 \ M_{\text{sun}} \)

Small black holes are more extreme

If you fall in then density \( \rightarrow \infty \) as you fall to the centre
What is a black hole?

- Escape velocity \( V = (2GM/R)^{1/2} = c \)
- \( R = 2GM/c^2 \)
- Not solid objects -- density \( \propto M/R^3 \propto 1/M^2 \)
- Simplest objects in the universe
- Only external observable properties are: mass, charge, and angular momentum
- Things go in. Nothing comes out.
- Form when massive stars die (if \( M > 3M_{\text{sun}} \)) and power quasars and galactic nuclei when \( M \approx (10^6-10^9)M_{\text{sun}} \)
- Can form in the early universe with all masses (even \( M \ll 3M_{\text{sun}} \))
Kerr Rotating Black Hole (1963)

\[ R = \frac{1}{2} \left\{ R_s + \sqrt{(R_s^2 - 4a^2)} \right\} \]

where \( R_s \equiv 2GM/c^2 \) and \( a \equiv J/Mc \) so \( J \leq GM^2/c \)
Black hole mechanics

Laws of Black Hole Mechanics

Zeroth: $g = \frac{GM}{R^2}$ constant on horizon

First: $\delta(Mc^2) = \delta A \frac{c^6}{32\pi G^2} + \delta W$

Second: $\delta A \geq 0$

Third: You can’t reduce $g$ to zero in a finite number of steps
Black hole thermodynamics

**Laws of Black Hole Mechanics**

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**Laws of Thermodynamics**

**T:** \( T = \text{constant} \)

**\( \delta E \):** \( \delta E = T \delta S + \delta W \)

**\( \delta S \):** \( \delta S \geq 0 \)

**You can’t reduce \( T \) to absolute zero in a finite number of steps**

**IDENTICAL IF AND ONLY IF**

Black hole entropy \( S = k c^3 A/4\hbar \) and temperature \( T = \hbar g/2\pi c k = \hbar c^3 /8\pi k GM \)
Links relativity (c), quantum (h), thermal (k) and gravitational (G) physics

\[ S = \frac{kc^3}{A/4G\hbar} = \left(\frac{kc^3}{4G\hbar}\right) \times 4\pi \left(\frac{2GM}{c^2}\right)^2 \]

\[ S = 4\pi k \frac{GM^2}{\hbar c} = 4\pi k \left(\frac{M}{m_{pl}}\right)^2 \]

\[ T = \frac{\hbar c^3}{8\pi kGM} \rightarrow \infty \text{ as } M \rightarrow 0 \]

Black hole evaporation and explosions
Hawking, 1975

dE/dt = d(Mc^2)/dt = area \times aT^4 \times c \propto M^2 \times M^{-4} \text{ so Lifetime } = \int dt \propto \int M^2 dM \propto M^3
Life time = 10 billion yrs \times (M/10^{14} \text{ gm})^3
Quantum black holes are black bodies

\[ \Delta E \Delta t \sim \hbar/2 \text{ and } \Delta E \sim E \sim kT_{bh} \]
\[ \Delta t \sim R_s/c \sim 2GM/c^3 \]
\[ T_{bh} \sim \hbar c^3/kGM \]
Evaporation of Black Holes (1974)

Lifetime = $14(M/10^{14} \text{ gm})^3$ billion yrs

Gamma rays
Radio bursts

Where does all the information go?
Further reading

- J. Gleick *The Information: a natural history of information theory*, Pantheon Books
- H.S. Leff and A.F. Rex, (editors), *Maxwell's Demon: Entropy, information, computing*. This volume contains reprints of all the key articles, including those by Maxwell, Szilard, Landauer and Bennett, together with a very nice overview of the history and key insights.
- S. Carlip, Black Hole Thermodynamics, online at https://arxiv.org/abs/1410.1486
- V.P. Frolov and A. Zelnikov, Introduction to Black Hole Physics, Oxford UP, (2011)