Concepts in Theoretical Physics

Lecture 6: General Relativity

John D Barrow

‘Einstein’s theory of relativity is probably the greatest synthetic achievement of the human intellect up to the present time’

Bertrand Russell (1955)

Newton’s Laws Are Un-Copernican

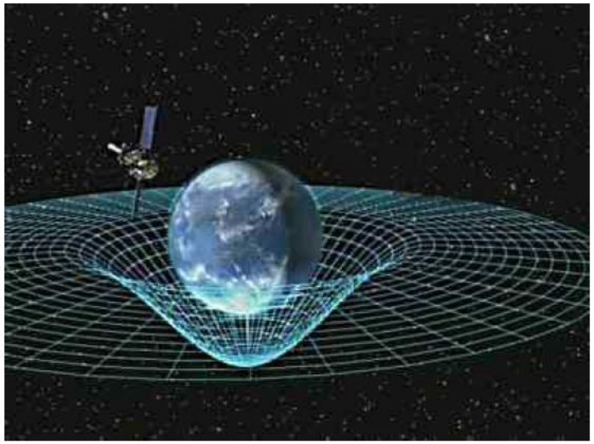
Newton’s laws only hold for special observers
who don’t rotate or
Accelerate with respect to the distant fixed stars

There are special classes of observers for whom the laws of Nature look simpler!!

Einstein’s equations are the same for all observers

The idea of general relativity

Space and time are not fixed and absolute



Mass and energy determine the geometry of space and the rate of flow of time locally

Einstein's eqns: {Geometry} = {mass energy content}
They have the same form under any coordinate transform

$$ds^2 = \Sigma\, g_{\mu\nu}(t, \mathbf{x}) dx^\mu dx^\nu$$ (sum over $\mu,\nu = 0,1,2,3$)

'Generalises' the metric of 'special' relativity in (t,x,y,z) coords:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

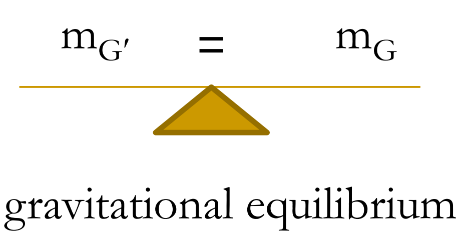
Eqns of motion determines the motions of masses and light on this curved geometry given by stationary action principle

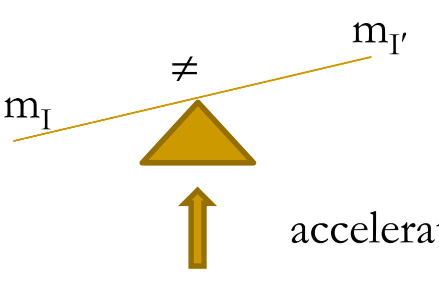
$$\delta \int ds = 0$$

"Matter tells space how to curve. Space tells matter how to move"

Inertial and Gravitational Mass

- Mass arises in two different formulae, both due to Newton
  - Gravitational Mass: $F = -Gm_G M_G / r^2$
  - Inertial Mass: $F = m_I a$
- Yet the meaning of mass in these two formulae is very different.
- We should really distinguish between the two masses by calling them something different.





They are measured to be the same to at least 1 part in 10^{12}

The Universality of Free-fall

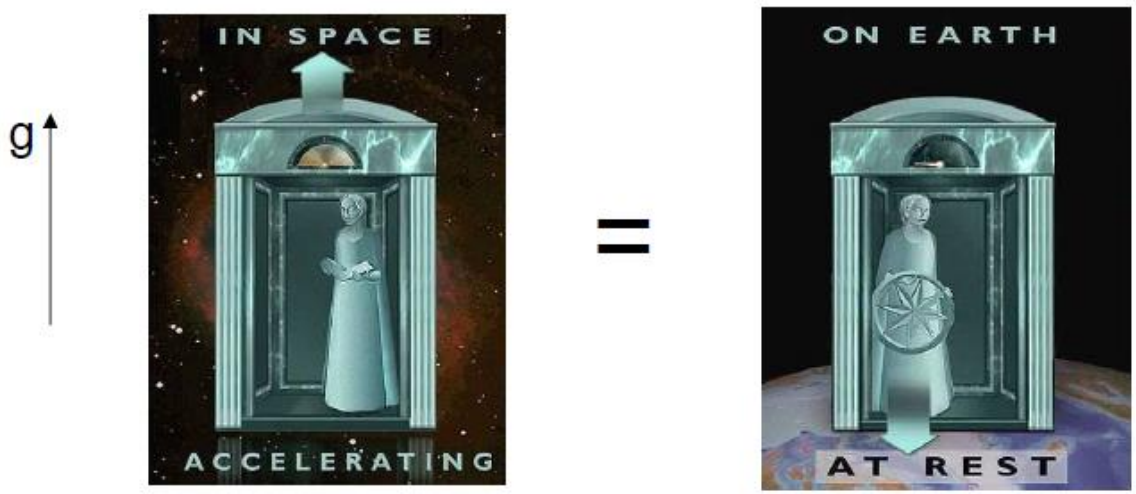
- The equality of inertial and gravitational mass is responsible for the well known fact that objects with different mass fall at the same speed under gravity.
- According to legend, this was demonstrated by Galileo dropping farm animals from the leaning tower of Pisa.
- But is there a deeper reason why the gravitational force is proportional to the inertial mass?

Ficticious forces

- There are two other forces which are also proportional to the inertial mass. These are
  - Centrifugal Force: $F = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$
  - Coriolis Force: $F = -2m\vec{\omega} \times \dot{\vec{r}}$
- But in both of these cases, we understand very well why the force is proportional to the inertial mass, m. It follows because these are “fictitious forces”, arising in a non-inertial frame. (In this case, one that is rotating with frequency $\vec{\omega}$)
- Could gravity also be a fictitious force, arising only because we are in a non-inertial frame? The answer, of course, is yes!

Einstein’s Principle of Equivalence

- Locally, it is impossible to distinguish between a gravitational field, and acceleration



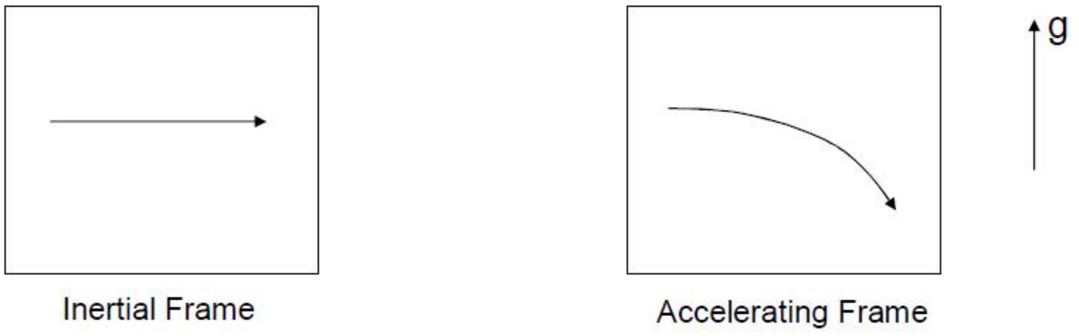
- Said another way:

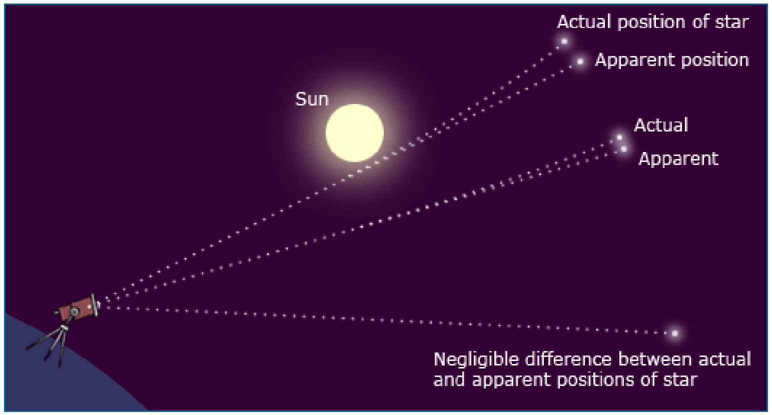
“The gravitational field has only a relative existence... Because for an observer freely falling from the roof of a house - at least in his immediate surroundings - there exists no gravitational field”.

A. Einstein

Predicting the bending of light

The first prediction is simple: light bends in a gravitational field

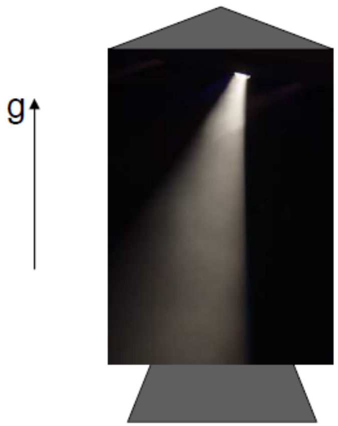




Maximum 'bending' occurs closest to the centre of a gravitational 'lens'
Not like an optical lens

Gravitational Redshift

- The second prediction again involves light. It undergoes a redshift in a gravitational field.



- The rocket has height h. It starts from rest, and moves with constant acceleration g.
- Light emitted from the top of the rocket is received below. By this time, the rocket is travelling at speed

$$v = gt = gh/c$$

- This gives rise to the Doppler effect. (Neglecting relativistic effects).

$$\nu' = \nu\left(1 + \frac{v}{c}\right) = \nu\left(1 + \frac{gh}{c^2}\right)$$

Received frequency

Emitted frequency

Gravity shifts light frequencies

- By the equivalence principle, the same effect must be observed in a gravitational field.
- The gravitational field Φ is defined to give rise to acceleration

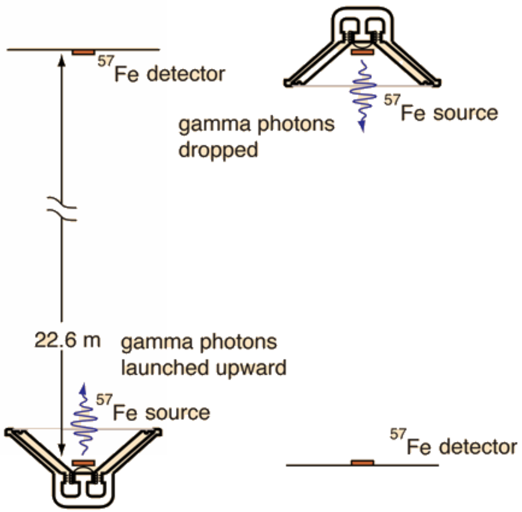
$$\vec{a} = -\nabla\Phi$$

- For constant gravitational acceleration, $\Phi = -gh$
- This gives the formula for gravitational redshift:

$$\nu' = \nu\left(1 - \frac{\Phi}{c^2}\right)$$

- Inverting, we get the relationship between the periods of light

$$T' = T\left(1 - \frac{\Phi}{c^2}\right)^{-1} \approx T\left(1 + \frac{\Phi}{c^2}\right)$$



Pound & Rebka, 1959

Spacetime Intervals

- Since time slows down in a gravitational field, we can think of writing this in a way familiar from special relativity

$$(dt')^2 = dt^2 \left(1 + \frac{\Phi}{c^2}\right)^2 \approx dt^2 \left(1 + \frac{2\Phi}{c^2}\right)$$

- Although this formula was derived under the assumption of constant acceleration, it actually holds for arbitrary gravitational potential $\Phi(\vec{x})$
- E.g. for a point particle of mass M, we have $\Phi = -GM/r$
- This means, that time runs slower on Earth than in space. If you choose to spend your life on the ground floor, and never climb the stairs, you live longer by a couple of microseconds.

$$(dt')^2 = dt^2 \left(1 - \frac{2GM}{rc^2}\right)$$

The metric of spacetime geometry

- We can write down a metric, akin to the Minkowski metric that you've met in special relativity. We've already seen that the time component should look like:

$$ds^2 = \left(1 + \frac{\Phi(x)}{c^2}\right) c^2 dt^2 + \ldots$$

- But, under Lorentz transformations, time and space mix together. So the space part should also vary with space. This means that measuring sticks contract and expand at different points in space. In general, the metric of spacetime is a 4x4 matrix, whose components can be any function of t and x.

$$ds^2 = g_{\mu\nu}(t, x)\, dx^\mu dx^\nu$$

Sum over μ,ν = 0,1,2,3

- Einstein's field equations tell you how to calculate the metric, describing how spacetime bends, for a given distribution of matter.

Einstein's equations of general relativity

- Aim to have 2nd order pdes describing gravity that are tensor eqns:
- {Spacetime geometry} = {mass-energy and motion}
- Tensors describe 4-d geometry of spacetime: Ricci $R_{\mu\nu}$, $R \equiv$ trace of $R_{\mu\nu}$, and $g_{\mu\nu}$
- General linear combination of rank-2 tensors (2nd derivatives of $g_{\mu\nu}$):

$$R_{\mu\nu} + ARg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \equiv \kappa\rho u_\mu u_\nu$$

A, Λ and κ constants, u_μ is the 4-velocity, ρ the density

- Tensor $T_{\mu\nu}$ chosen so $\nabla^\mu T_{\mu\nu} = 0 \leftrightarrow$ conservation of energy + momentum
- $\nabla^\mu g_{\mu\nu} = 0$ always and $\nabla^\mu(R_{\mu\nu} + ARg_{\mu\nu}) = 0$ iff $A = -\frac{1}{2}$ is a theorem of differential geometry

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \leftarrow \text{Einstein's eqns}$$

Weak gravity limit with $\Lambda = 0$: $R_{00} = \frac{1}{2}\nabla^2 g_{00} = \frac{1}{2}\nabla^2(1 + 2\Phi/c^2) = \frac{1}{2}\kappa\rho c^2$

$$\nabla^2\Phi = \tfrac{1}{2}\kappa\rho c^4$$

This is Poisson's eqn for Newtonian gravity if $\kappa = 8\pi G/c^4$

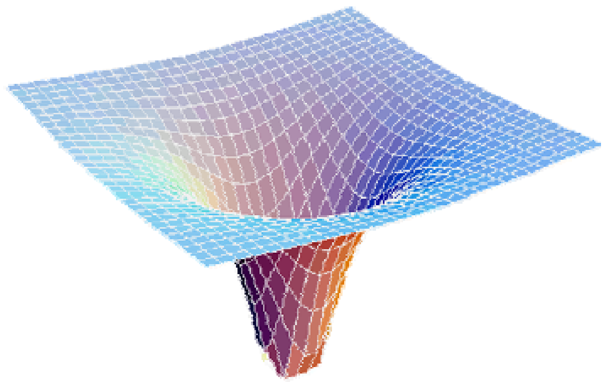
$$\nabla^2\Phi = 4\pi G\rho$$

- If $\Lambda \neq 0$ then $\nabla^2\Phi + \Lambda = 4\pi G\rho$

What is this strange new constant Λ ? (see next lecture)

Karl Schwarzschild's solution

- The metric for spacetime around a black hole, or outside a spherical star, is the Schwarzchild solution



$$ds^2 = c^2dt^2 (1 - 2GM/rc^2) – dr^2 (1 - 2GM/rc^2)^{-1} – r^2(d\theta^2 + \sin^2\theta\ d\phi^2)$$

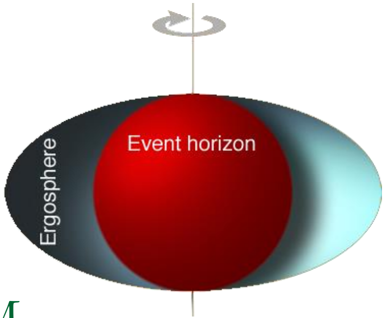
Accurately describes the gravitational field of the Sun.
Equations of motion give the orbits of the planets

Kerr's rotating black hole solution (1963)

$$ds^2 = -dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta\, d\phi^2 + \frac{2GMr}{\rho^2}(a\sin^2\theta\, d\phi - dt)^2 ,$$

$$\Delta(r) = r^2 - 2GMr + a^2 ,$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta .$$



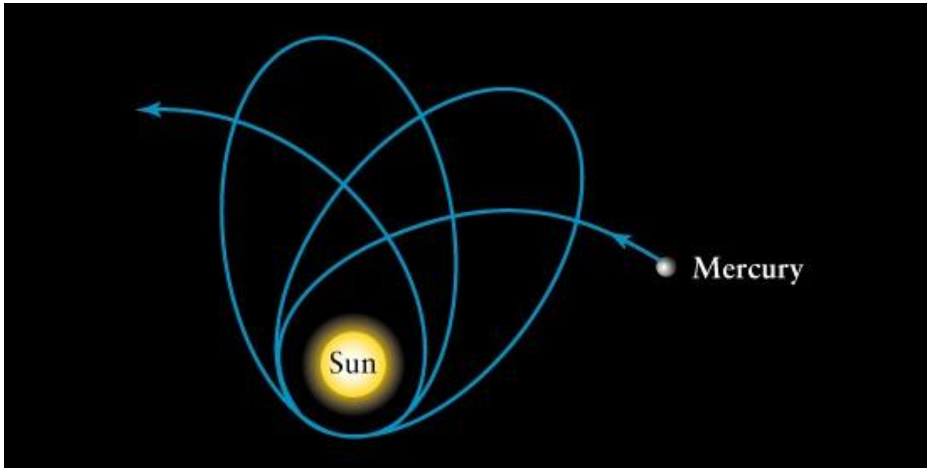
This is the spacetime metric around a black hole of mass M and angular momentum per unit mass $a = J/M$ in units where $c = 1$. Ergosphere has $E < 0$ orbits so can extract rotation energy.

The coordinates are 'Boyer-Lindquist' coordinates (t, r, θ, ϕ)

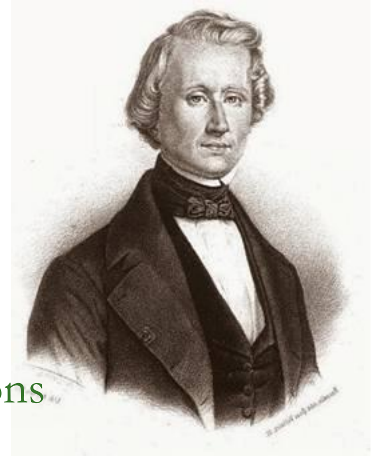
Cartesians $x = (r^2 + a^2)^{1/2}\sin\theta\cos\phi$, $y = (r^2 + a^2)^{1/2}\sin\theta\sin\phi$, $z = r\cos\theta$

When $a \to 0$ it becomes Schwarzschild's metric

Perihelion precession of the planets



1859 observations



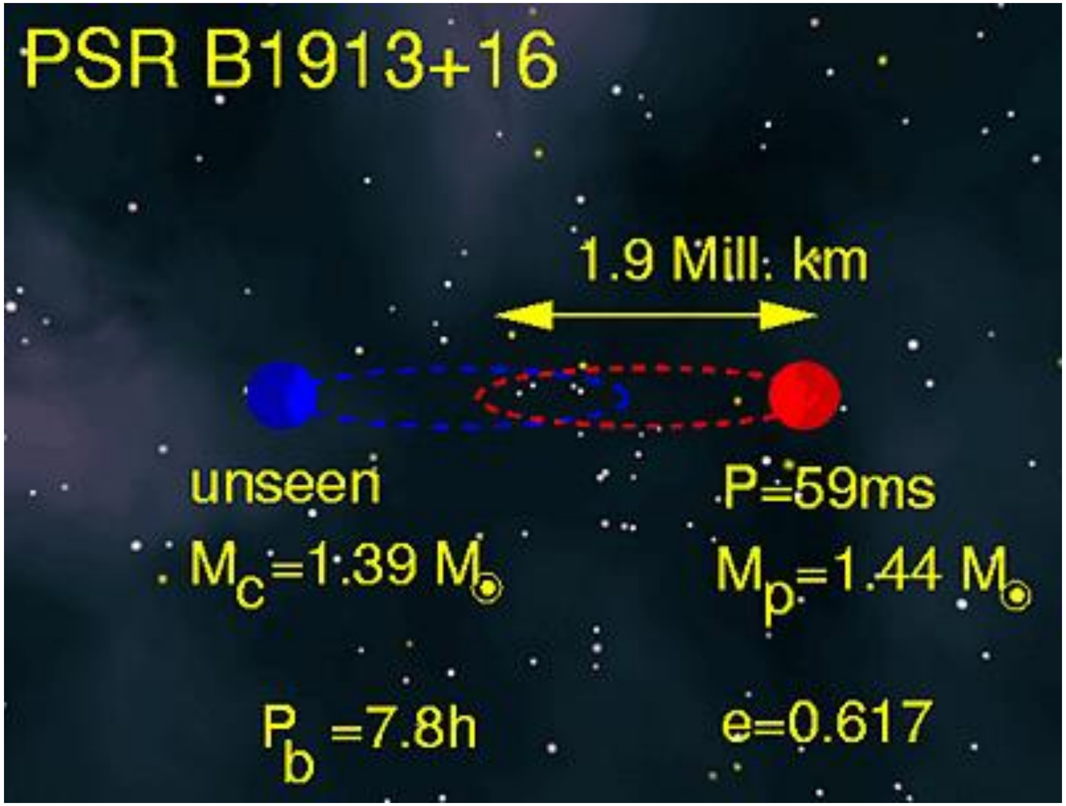
Urbain le Verrier

Planet	Newtonian Value	Observed Value	Difference	Prediction GTR
Mercury	532.08	575.19	43.11 ± 0.45	43.03
Venus	13.2	21.6	8.4 ± 4.8	8.6
Earth	1165	1170	5 ± 1.2	3.8

Perihelion advance in seconds of arc per century

Current confirmation of general relativity to a few parts in 10^5

Hulse-Taylor Binary Pulsar (1974)

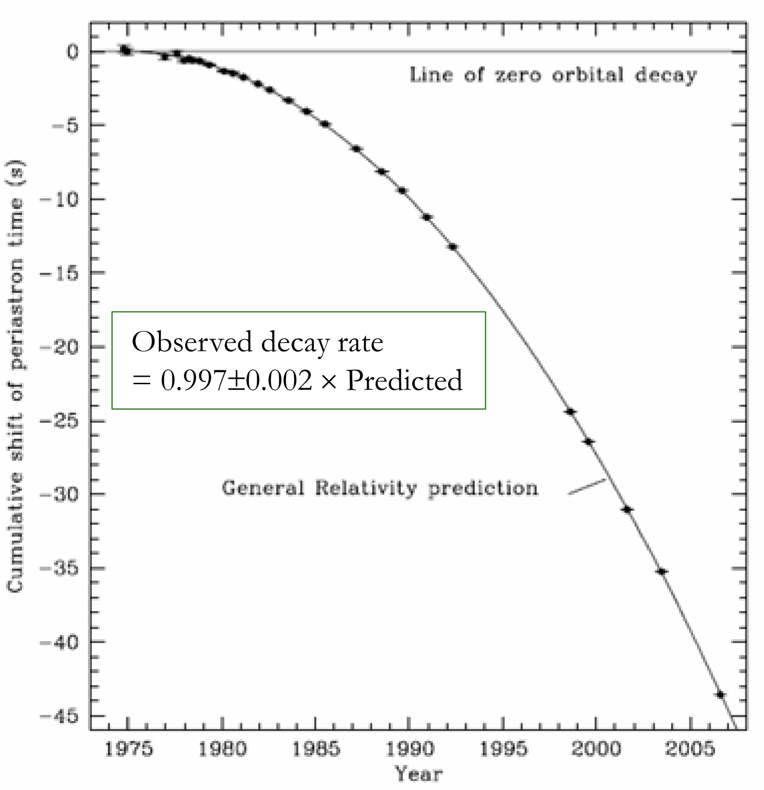


21,000 light yrs away
in Aquila

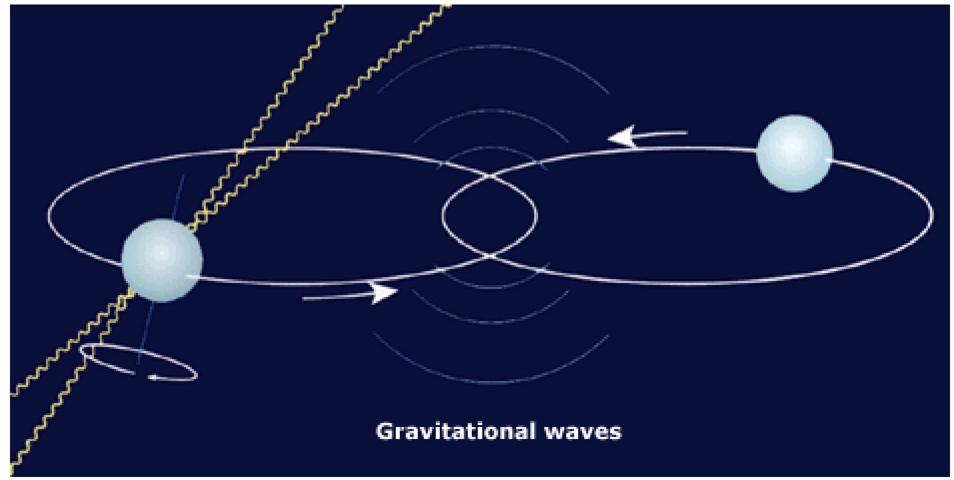
Pulses tracked to
15 mu sec

Perihelion advance is 4.2 degrees per year

Binary Pulsar's Orbital Decay



Weisberg et al, 1981

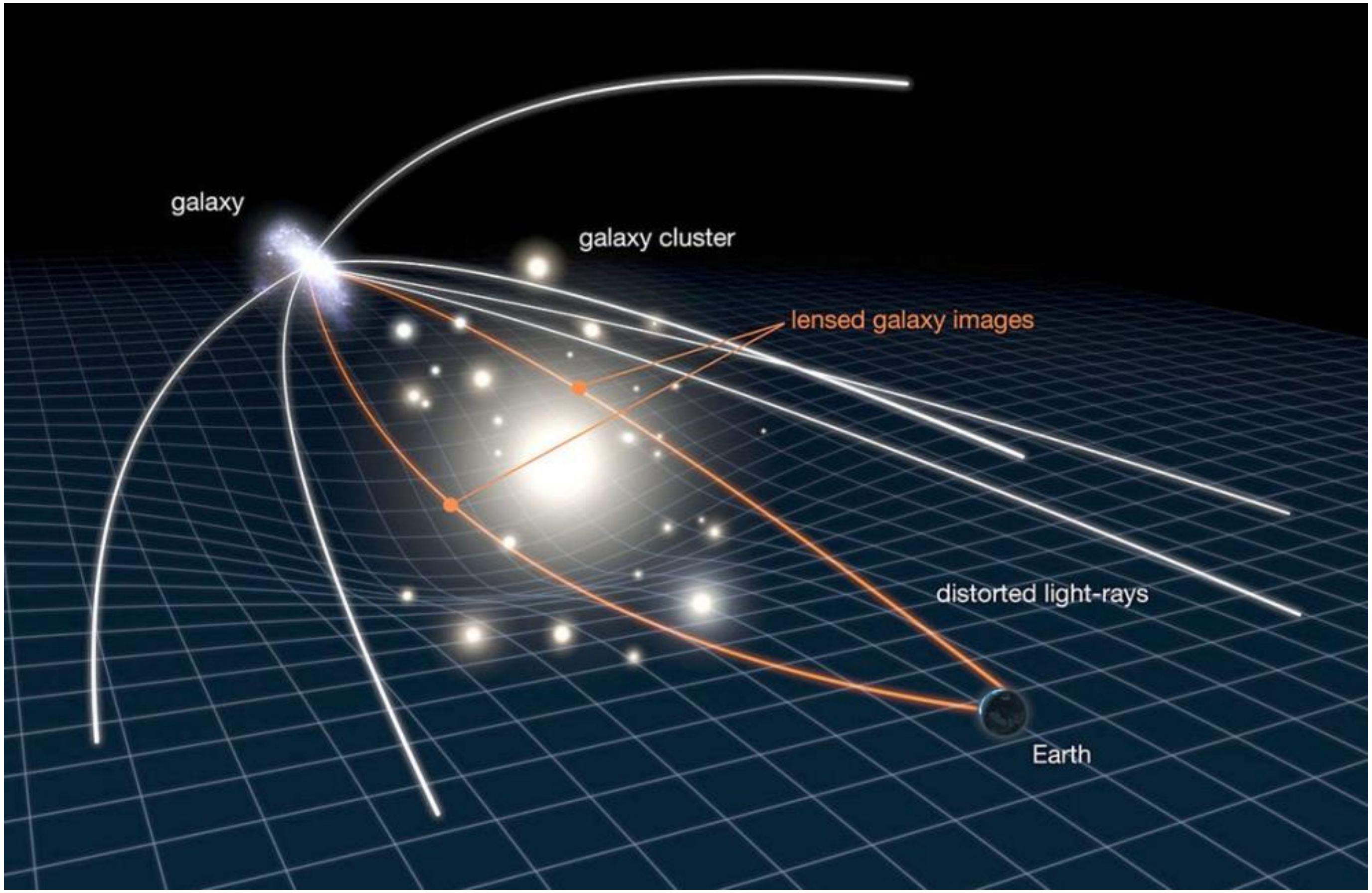


PSR B1913+16

Hulse & Taylor, 1974

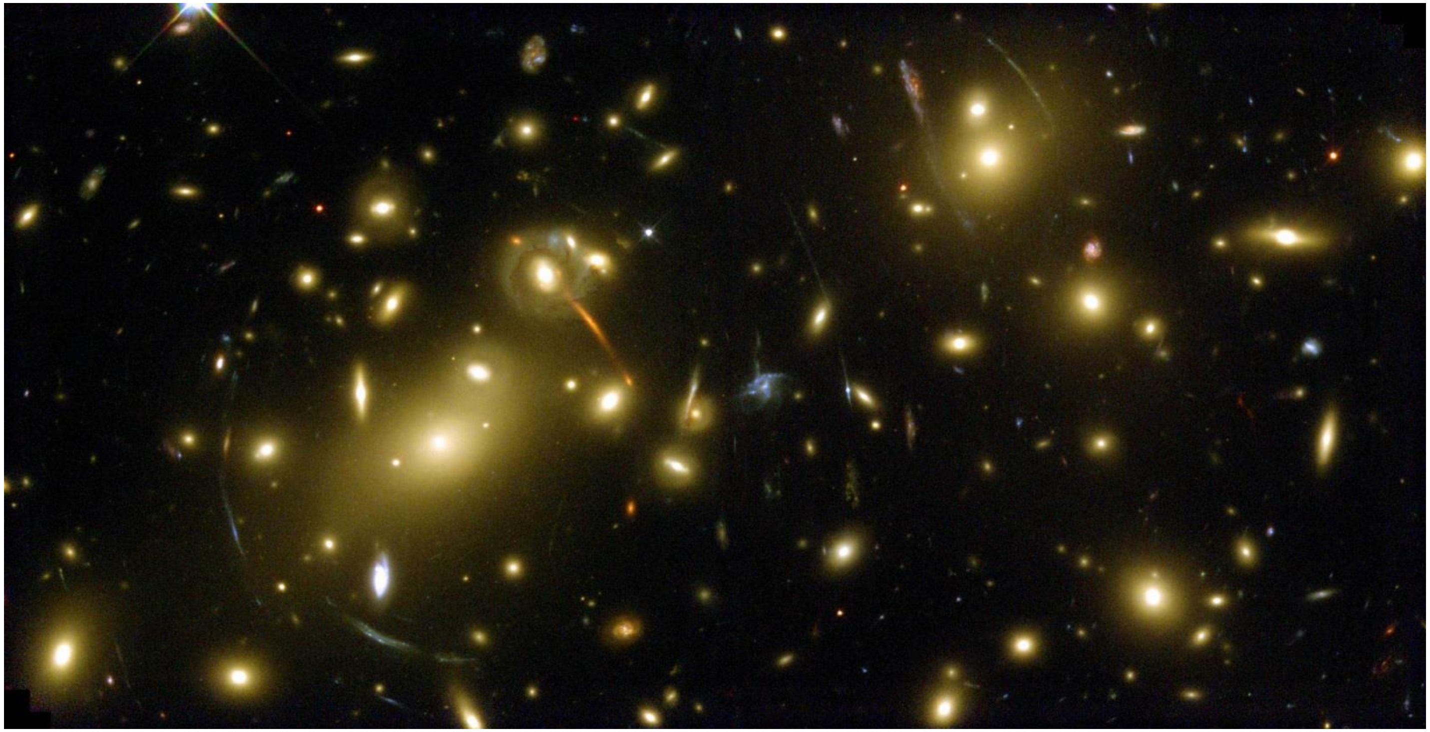
Consistent with gravitational radiation emission predictions of general relativity

Detecting 'dark matter' by lensing



irfu.cea.fr

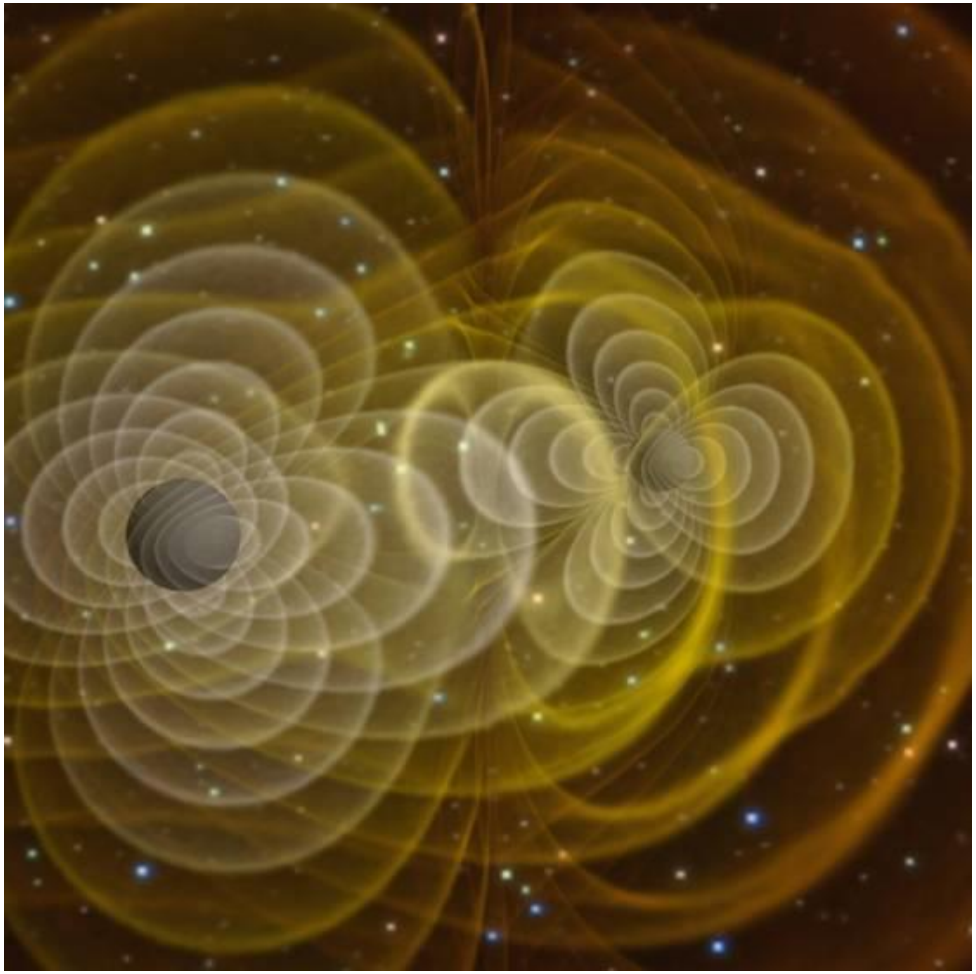
Gravitational lensing of galaxies

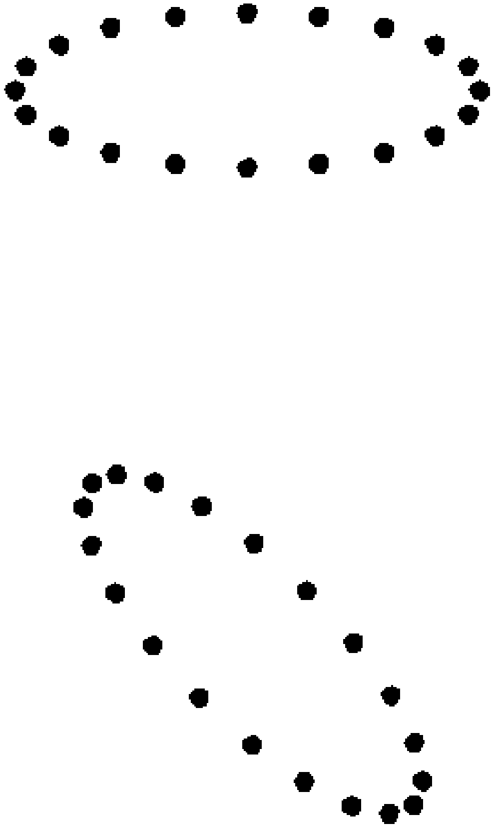


Galaxy Cluster Abell 1689

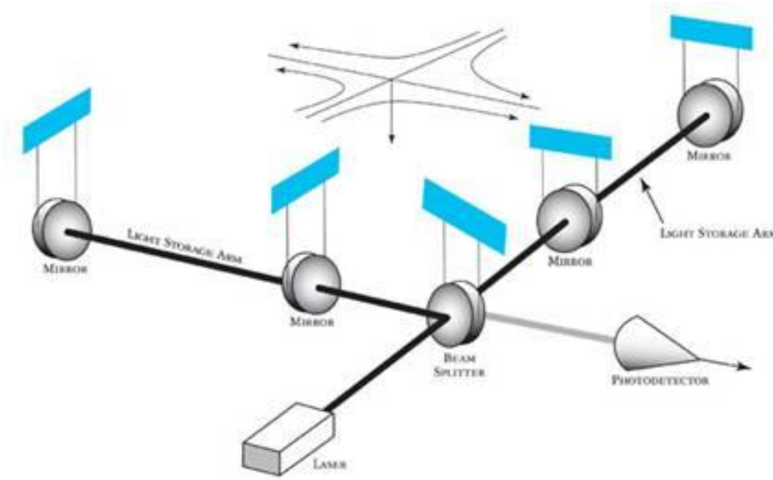
$$M_{dark} \approx 10 \times M_{luminous}$$

Gravitational Waves



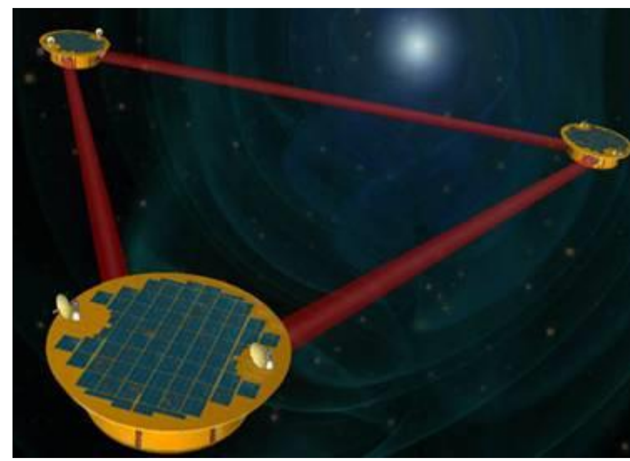


LIGO Gravitational wave interferometer



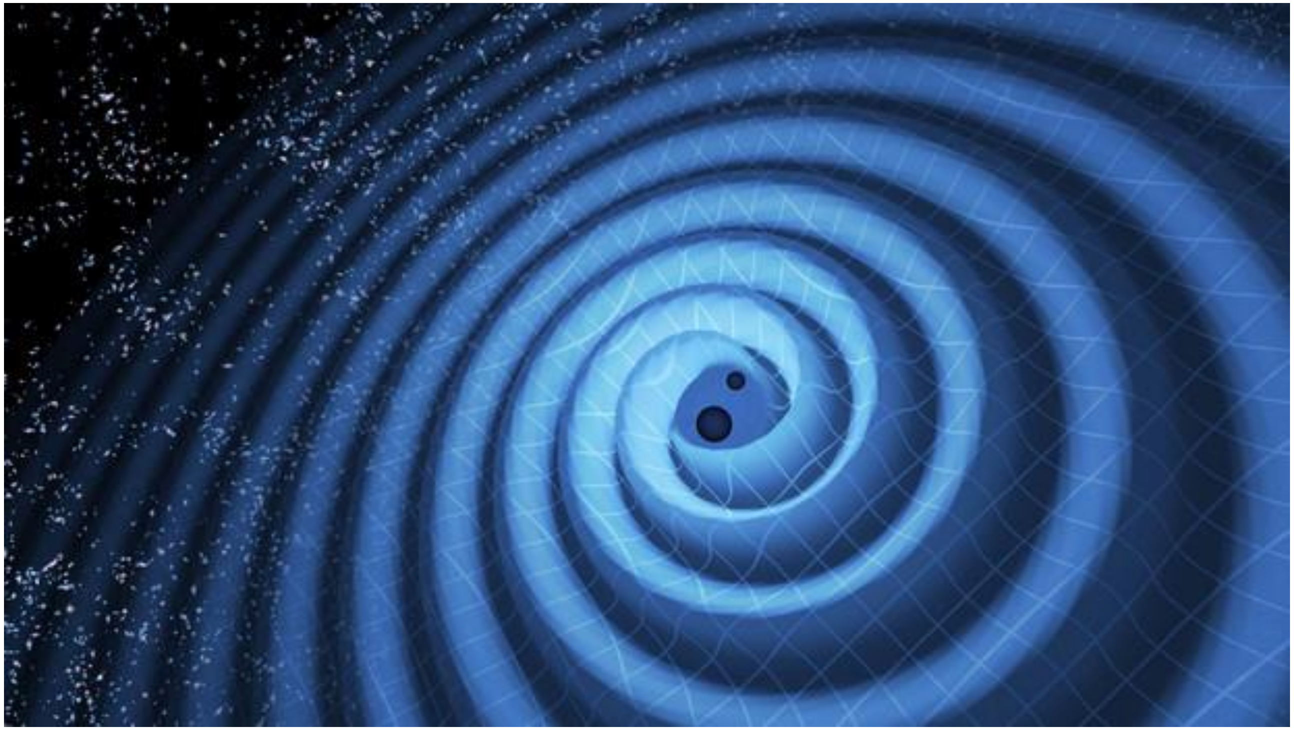
Basic interferometer

Hanford, WA



Now joined in June 2017 by the VIRGO interferometer near Pisa, Italy

Inspiralling of two spinning black holes

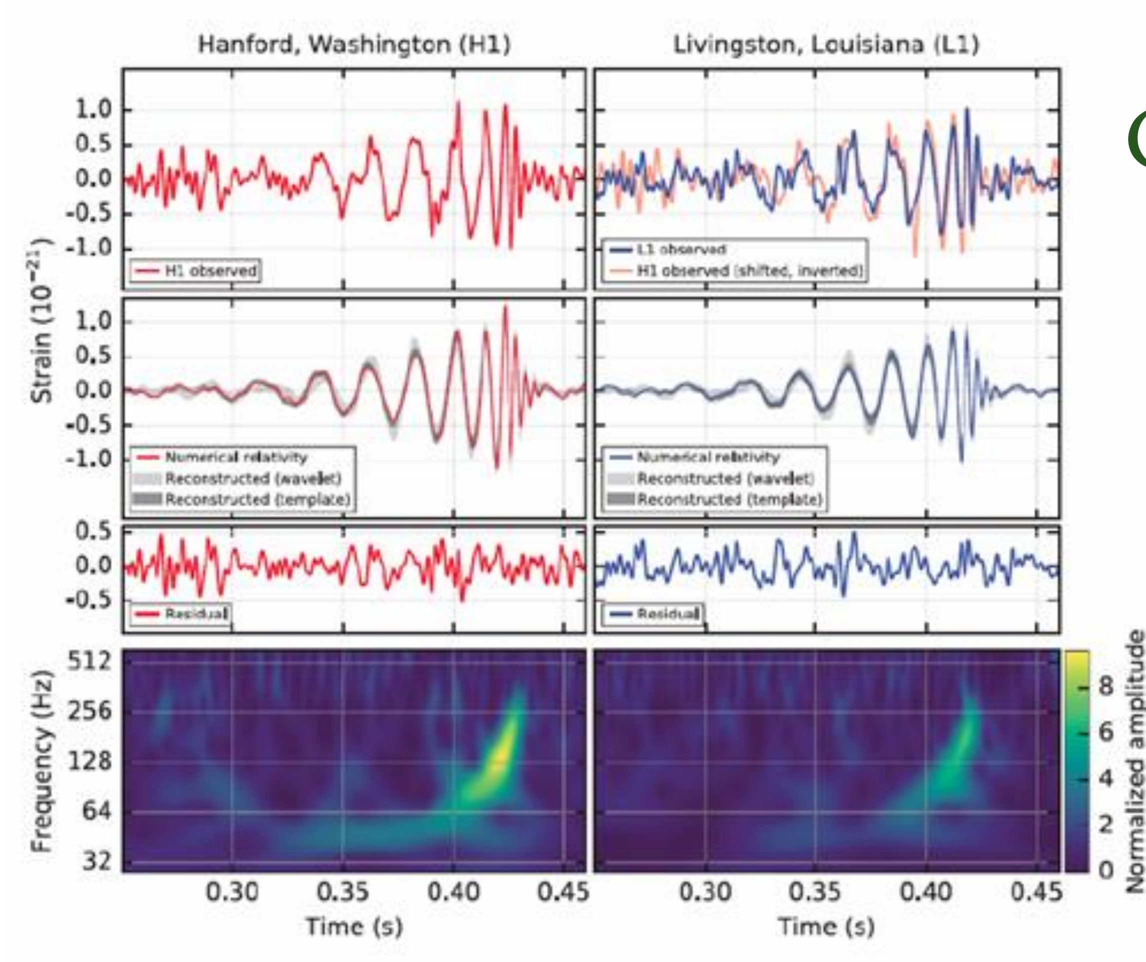


LIGO/T. Pyle

Spinning black holes merge (2015)

14 M_{sun} + 8 M_{sun} → 21 M_{sun} + lost energy (including gravity waves)

First Detection of Gravitational Waves

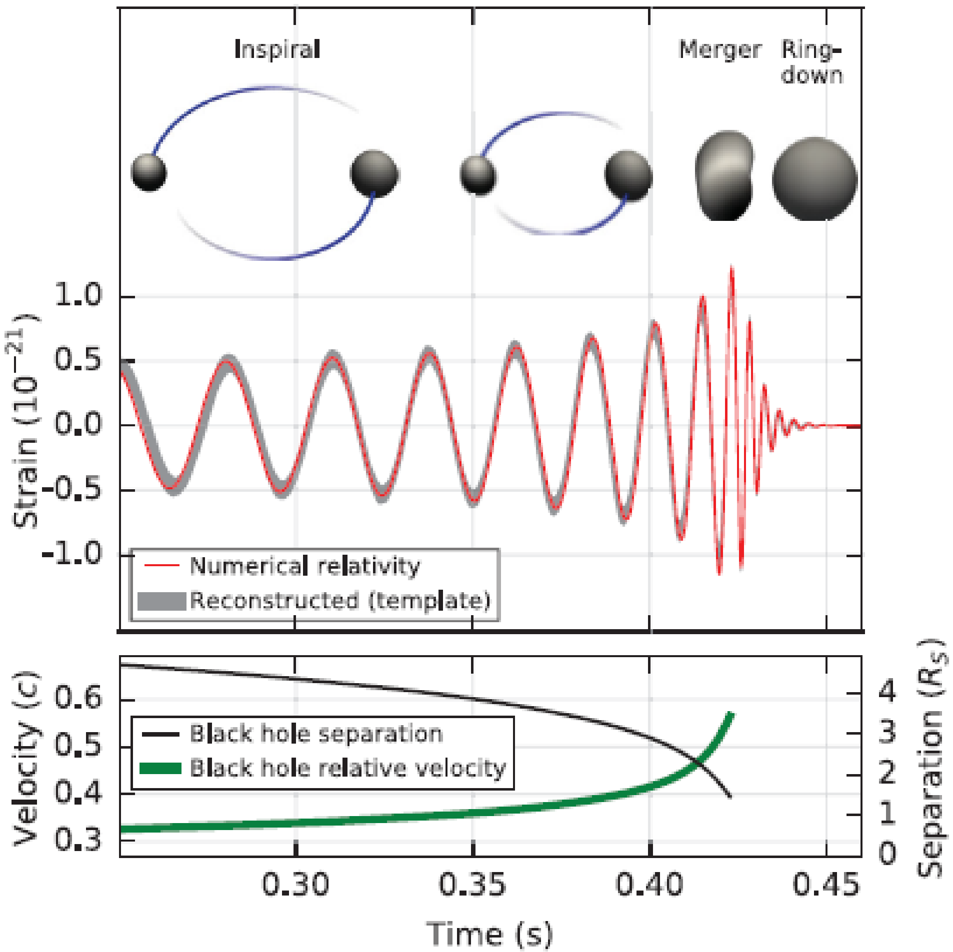


GW 150914

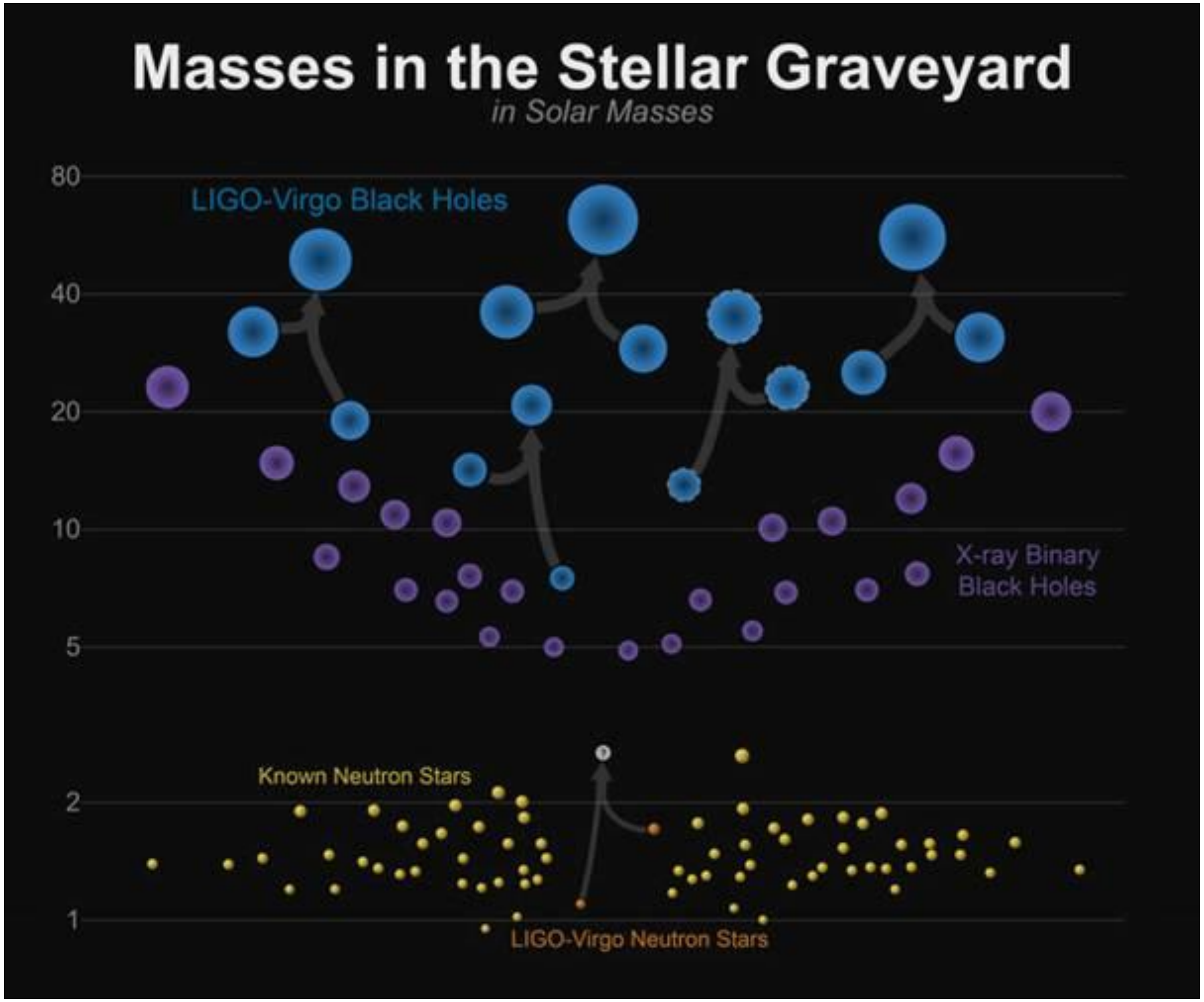
26 Dec 2015

See https://arxiv.org/pdf/1609.09349

The signal profile



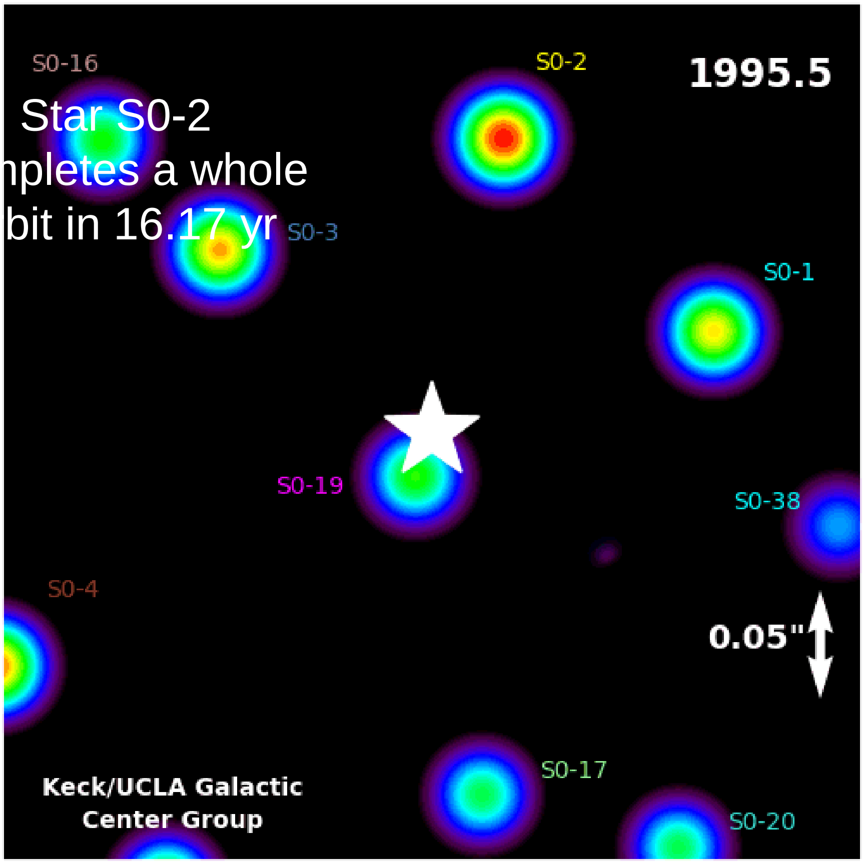
Black Holes



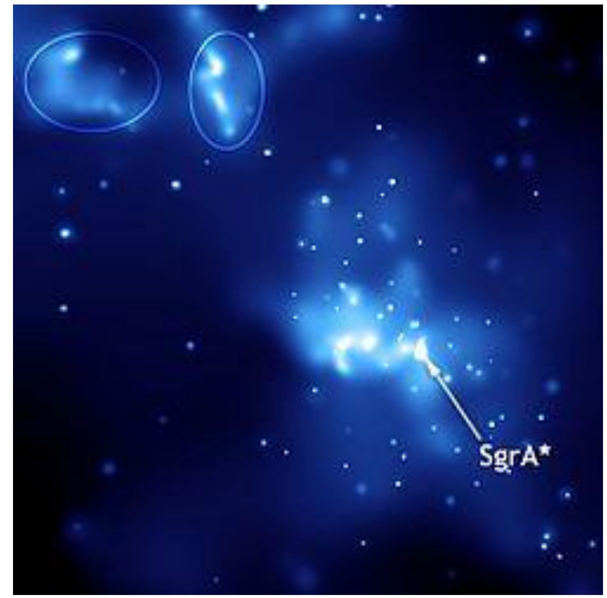
Black holes
Blue = by GWs
Purple = by photons

Neutron stars
Yellow = by photons
Orange = by GWs

Black Hole at the Centre of the Milky Way

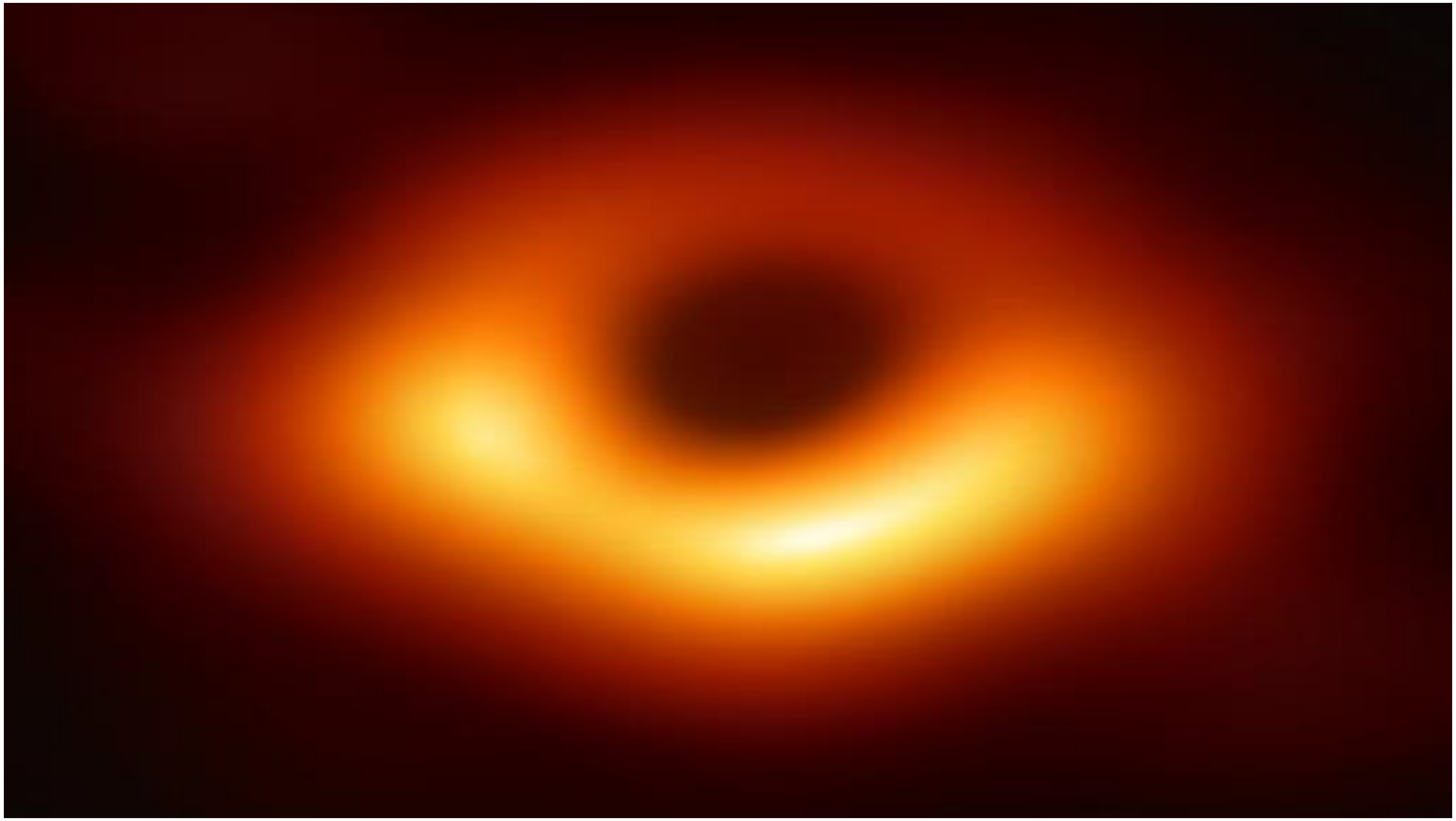


Speeds of stars up to 3 million mph!



Mass = 4 x 10^6 solar masses

Black Hole at the centre of Galaxy M87



Mass = 6.5×10^9 solar masses

53.5 million light years away

Further reading

- A. Einstein The Meaning of General Relativity, Methuen, 1955, new pbk edn by Routledge 2003
- T-P. Cheng, Relativity, Gravitation and Cosmology: a basic introduction, Oxford UP (2008)
- L. Landau and E.M. Lifshitz, The Classical Theory of Fields, 4th rev. edn., Pergamon, (1974)
- C Will, *Was Einstein Right?,* 2nd edn, Basic Books, (1993)
- V.P. Frolov and A. Zelnikov, Introduction to Black Hole Physics, Oxford UP, (2011)
- LIGO and Virgo Collaborations (954 authors!), The basic physics of the binary black hole merger GW 150914, online at https://arxiv.org/abs/1608.01940
- K. Thorne and R. Blandford, *Modern Classical Physics*, Princeton UP (2017), chapters 25-27 cover general relativity, black holes and gravitational waves.