Mathematical Tripos Part IA | **Michaelmas Term 2023**
Vectors and Matrices | **Prof. N. Peake**

Example Sheet 1

1. Let D be the interior of the circle $|z-1-i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in D$ then

$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1 \,.$$

Obtain the same result geometrically [start by considering the line through the centre of the circle and the point 3].

2. Given $|z| = 1$ and $\arg z = \theta$, find both algebraically and geometrically the modulus-argument forms of

$$\text{(i)} \quad 1 + z, \qquad \text{(ii)} \quad 1 - z \,.$$

Show that the locus of w as z varies with $|z| = 1$, where w is given by

$$w^2 = \frac{1-z}{1+z} \,,$$

is a pair of straight lines.

3. Consider a triangle in the complex plane with vertices at 0, z_1 and z_2. Write down an expression for the general point on the median through z_1, and a similar expression for the general point on the median through z_2. Show that the three medians of the triangle are concurrent.

4. Express

$$I = \frac{z^5 - 1}{z - 1}$$

as a polynomial in z. By considering the complex fifth root of unity ω, obtain the four factors of I linear in z. Hence write I as the product of two real quadratic factors. By considering the term in z^2 in the identity so obtained for I, show that

$$4 \cos\frac{\pi}{5} \, \sin\frac{\pi}{10} = 1 \,.$$

5. Find all complex numbers z that satisfy $\sin z = 2$.

6. (a) Let $z, a, b \in \mathbb{C}$ $(a \neq b)$ correspond to points P, A, B in the Argand diagram. Let C_λ be the locus of P defined by

$$|PA|/|PB| = \lambda \,,$$

where λ is a fixed real positive constant. Show that C_λ is a circle if $\lambda \neq 1$, and find its centre and radius. What happens if $\lambda = 1$?

(b) For the case $a = -b = p,\ p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that

$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$

is a circle passing through A and B with its centre on the perpendicular bisector of AB.
Show that the circles C_λ and S_μ intersect orthogonally for all λ, μ.

7. Show by vector methods that the altitudes of a triangle are concurrent.
[*Hint:* let the altitudes AD, BE of $\triangle ABC$ meet at H, and show that CH is perpendicular to AB.]

8. In three dimensions, the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{a}$ satisfy

$$\mathbf{x} + \mathbf{y}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{a} \,.$$

Show that

$$(\mathbf{x} \cdot \mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2 + |\mathbf{y}|^2} \,.$$

Use an inequality involving $\mathbf{x}\cdot\mathbf{y}$ and the lengths of $\mathbf{x}$ and $\mathbf{y}$ to deduce that

$$|\mathbf{x}|(1+|\mathbf{y}|^2) \geqslant |\mathbf{a}| \geqslant |\mathbf{x}|\ .$$

Explain the circumstances in which either of the inequalities above become equalities, and describe the relation between $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{a}$ in these circumstances.

9. (a) In $\triangle ABC$, let $\overrightarrow{AB}=\mathbf{u}$, $\overrightarrow{BC}=\mathbf{v}$ and $\overrightarrow{CA}=\mathbf{w}$. Show that

$$\mathbf{u}\times\mathbf{v}=\mathbf{v}\times\mathbf{w}=\mathbf{w}\times\mathbf{u}\,,$$

and hence obtain the sine rule for $\triangle ABC$.

(b) Given any three vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ such that

$$\mathbf{p}\times\mathbf{q}=\mathbf{q}\times\mathbf{r}=\mathbf{r}\times\mathbf{p}\,,$$

and $|\mathbf{p}\times\mathbf{q}|\neq 0$, show that

$$\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}\,.$$

10. Interpret geometrically the equations

$$\mathbf{r} \;=\; (1-\lambda)\mathbf{a} \;+\; \lambda\mathbf{b}\,,$$

and

$$\mathbf{r} \;=\; (1-\lambda-\mu)\mathbf{a} \;+\; \lambda\mathbf{b} \;+\; \mu\mathbf{c}\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of points in three dimensions and λ, μ are scalar parameters that take any real values. For each equation, obtain an equivalent form that does not involve parameters λ or μ. How do your equations behave when the points $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$ are interchanged?

11. (a) Using the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, show that

$$\begin{array}{rl}\text{(i)} & (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) \;=\; (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,,\\ \text{(ii)} & \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \;+\; \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) \;+\; \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) \;=\; \mathbf{0}\,.\end{array}$$

Relate the case when $\mathbf{c}=\mathbf{a}$ and $\mathbf{d}=\mathbf{b}$ in (i) to a well-known trigonometric identity.

Evaluate $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ in two ways and compare the results to find an explicit linear combination of the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ that is zero.

(b) Given that $[\mathbf{a},\mathbf{b},\mathbf{c}]\equiv\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, show that

$$[\,\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{c},\,\mathbf{c}\times\mathbf{a}\,] \;=\; [\mathbf{a},\mathbf{b},\mathbf{c}]^2.$$

12. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be fixed vectors in three dimensions. For each of the following equations, find all solutions for $\mathbf{r}$:

$$\text{(i)}\ \ \mathbf{r}\;+\;\mathbf{r}\times\mathbf{d}=\mathbf{c}\,; \qquad \text{(ii)}\ \ \mathbf{r}+(\mathbf{r}\cdot\mathbf{a})\,\mathbf{b}=\mathbf{c}\,.$$

[In (ii), consider separately the cases $\mathbf{a}\cdot\mathbf{b}\neq -1$ and $\mathbf{a}\cdot\mathbf{b}=-1$.]

13. The vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are defined in terms of the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by

$$\begin{aligned}\mathbf{e}_r &= \cos\phi\sin\theta\,\mathbf{i}+\sin\phi\sin\theta\,\mathbf{j}+\cos\theta\,\mathbf{k}\,,\\ \mathbf{e}_\theta &= \cos\phi\cos\theta\,\mathbf{i}+\sin\phi\cos\theta\,\mathbf{j}-\sin\theta\,\mathbf{k}\,,\\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i}+\cos\phi\,\mathbf{j}\end{aligned}$$

where θ and ϕ are real numbers. Show, as efficiently as possible, that $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are an orthonormal right-handed set.

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