Example Sheet 1

1. Let $D$ be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in D$ then
   \[ \sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1. \]
   Obtain the same result geometrically [start by considering the line through the centre of the circle and the point 3].

2. Given $|z| = 1$ and arg $z = \theta$, find both algebraically and geometrically the modulus-argument forms of
   \( i) \ 1 + z, \quad (ii) \ 1 - z. \)
   Show that the locus of $w$ as $z$ varies with $|z| = 1$, where $w$ is given by
   \[ w^2 = \frac{1 - z}{1 + z}, \]
   is a pair of straight lines.

3. Consider a triangle in the complex plane with vertices at 0, $z_1$ and $z_2$. Write down an expression for the general point on the median through $z_1$, and a similar expression for the general point on the median through $z_2$. Show that the three medians of the triangle are concurrent.

4. Express
   \[ I = \frac{z^5 - 1}{z - 1} \]
   as a polynomial in $z$. By considering the complex fifth root of unity $\omega$, obtain the four factors of $I$ linear in $z$. Hence write $I$ as the product of two real quadratic factors. By considering the term in $z^2$ in the identity so obtained for $I$, show that
   \[ 4 \cos \frac{\pi}{5} \sin \frac{\pi}{10} = 1. \]

5. Find all complex numbers $z$ that satisfy $\sin z = 2$.

6. (a) Let $z, a, b \in \mathbb{C}$ ($a \neq b$) correspond to points $P, A, B$ in the Argand diagram. Let $C_\lambda$ be the locus of $P$ defined by
   \[ |PA|/|PB| = \lambda, \]
   where $\lambda$ is a fixed real positive constant. Show that $C_\lambda$ is a circle if $\lambda \neq 1$, and find its centre and radius. What happens if $\lambda = 1$?
   (b) For the case $a = -b = p, \ p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that
   \[ S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\} \]
   is a circle passing through $A$ and $B$ with its centre on the perpendicular bisector of $AB$.
   Show that the circles $C_\lambda$ and $S_\mu$ intersect orthogonally for all $\lambda, \mu$.

7. Show by vector methods that the altitudes of a triangle are concurrent. [Hint: let the altitudes $AD, BE$ of $\triangle ABC$ meet at $H$, and show that $CH$ is perpendicular to $AB$.]

8. In three dimensions, the vectors $x, y$ and $a$ satisfy
   \[ x + y(x \cdot y) = a. \]
   Show that
   \[ (x \cdot y)^2 = \frac{|a|^2 - |x|^2}{2 + |y|^2}. \]
Use an inequality involving \( \mathbf{x} \cdot \mathbf{y} \) and the lengths of \( \mathbf{x} \) and \( \mathbf{y} \) to deduce that

\[
|\mathbf{x}|(1 + |\mathbf{y}|^2) \geq |\mathbf{a}| \geq |\mathbf{x}|.
\]

Explain the circumstances in which either of the inequalities above become equalities, and describe the relation between \( \mathbf{x}, \mathbf{y} \) and \( \mathbf{a} \) in these circumstances.

9. (a) In \( \triangle ABC \), let \( \overrightarrow{AB} = \mathbf{u} \), \( \overrightarrow{BC} = \mathbf{v} \) and \( \overrightarrow{CA} = \mathbf{w} \). Show that

\[
\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u},
\]

and hence obtain the sine rule for \( \triangle ABC \).

(b) Given any three vectors \( \mathbf{p}, \mathbf{q}, \mathbf{r} \) such that

\[
\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{r} = \mathbf{r} \times \mathbf{p},
\]

and \( |\mathbf{p} \times \mathbf{q}| \neq 0 \), show that

\[
\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}.
\]

10. Interpret geometrically the equations

\[
\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda \mathbf{b},
\]

and

\[
\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c},
\]

where \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are the position vectors of points in three dimensions and \( \lambda, \mu \) are scalar parameters that take any real values. For each equation, obtain an equivalent form that does not involve parameters \( \lambda \) or \( \mu \). How do your equations behave when the points \( \mathbf{a}, \mathbf{b} \) or \( \mathbf{c} \) are interchanged?

11. (a) Using the identity \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \), show that

\[
\begin{align*}
(i) & \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}), \\
(ii) & \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.
\end{align*}
\]

Relate the case when \( \mathbf{c} = \mathbf{a} \) and \( \mathbf{d} = \mathbf{b} \) in (i) to a well-known trigonometric identity. Evaluate \( (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) \) in two ways and compare the results to find an explicit linear combination of the four vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \) that is zero.

(b) Given that \( [\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] \), show that

\[
[a \times b, b \times c, c \times a] = [a, b, c]^2.
\]

12. Let \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \) be fixed vectors in three dimensions. For each of the following equations, find all solutions for \( \mathbf{r} \):

\[
\begin{align*}
(i) & \quad \mathbf{r} + \mathbf{r} \times \mathbf{d} = \mathbf{c}; \\
(ii) & \quad \mathbf{r} + (\mathbf{r} \cdot \mathbf{a})\mathbf{b} = \mathbf{c}.
\end{align*}
\]

[In (ii), consider separately the cases \( \mathbf{a} \cdot \mathbf{b} \neq -1 \) and \( \mathbf{a} \cdot \mathbf{b} = -1 \).]

13. The vectors \( \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi \) are defined in terms of the standard basis vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) by

\[
\begin{align*}
\mathbf{e}_r & = \cos \phi \sin \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \theta \mathbf{k}, \\
\mathbf{e}_\theta & = \cos \phi \cos \theta \mathbf{i} + \sin \phi \cos \theta \mathbf{j} - \sin \theta \mathbf{k}, \\
\mathbf{e}_\phi & = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j},
\end{align*}
\]

where \( \theta \) and \( \phi \) are real numbers. Show, as efficiently as possible, that \( \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi \) are an orthonormal right-handed set.

Comments to: J.M.Evans@damtp.cam.ac.uk