A1a  Vectors and Matrices: Example Sheet 1  Michaelmas 2014

A * denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are not necessarily harder than unstarred questions.

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1. Let $S$ be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in S$ then

$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1.$$ 

Obtain the same result geometrically by considering the line containing the centre of the circle and the point 3.

2. Given $|z| = 1$ and $\arg z = \theta$, find both algebraically and geometrically the modulus-argument forms of

(i) $1 + z$,  (ii) $1 - z$.

Show that the locus of $w$ as $z$ varies with $|z| = 1$, where $w$ is given by

$$w^2 = \left(\frac{1 - z}{1 + z}\right),$$

is a pair of straight lines.

3. Use complex numbers to show that the medians of a triangle are concurrent.

Hint: represent the vertices of the triangle by complex numbers $z_1$, $z_2$ and $z_3$ (or 0, $z_1$ and $z_2$ if you prefer), then write down equations for two of the medians and find their intersection.

*4. Express

$$I = \frac{z^5 - 1}{z - 1}$$

as a polynomial in $z$. By considering the complex fifth root of unity $\omega$, obtain the four factors of $I$ linear in $z$. Hence write $I$ as the product of two real quadratic factors. By considering the term in $z^2$ in the identity so obtained for $I$, show that

$$4 \cos \frac{\pi}{5} \sin \frac{\pi}{10} = 1.$$ 

5. Show the equation $\sin z = 2$ has infinitely many solutions.

6. (a) Let $z, a, b \in \mathbb{C} (a \neq b)$ correspond to the points $P, A, B$ of the Argand plane. Let $C_\lambda$ be the locus of $P$ defined by

$$PA/PB = \lambda,$$

where $\lambda$ is a fixed real positive constant. Show that $C_\lambda$ is a circle, unless $\lambda = 1$, and find its centre and radius. What if $\lambda = 1$?

*(b) For the case $a = -b = p$, $p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that the curve

$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$

is a circle passing through $A$ and $B$ with its centre on the perpendicular bisector of $AB$.

Show that the circles $C_\lambda$ and $S_\mu$ are orthogonal for all $\lambda, \mu$.

7. Show by vector methods that the altitudes of a triangle are concurrent.

Hint: let the altitudes $AD, BE$ of $\triangle ABC$ meet at $H$, and show that $CH$ is perpendicular to $AB$.

8. Given that vectors $x$ and $y$ satisfy

$$x + y(x \cdot y) = a,$$

for fixed vector $a$, show that

$$(x \cdot y)^2 = \frac{|a|^2 - |x|^2}{2 + |y|^2}.$$
Deduce using the Schwarz inequality (or otherwise) that

\[ |x|(1 + |y|^2) \geq |a| \geq |x| . \]

Explain the circumstances under which either of the inequalities can be replaced by equalities, and describe the relation between \( x, y \) and \( a \) in these circumstances.

9. (a) In \( \triangle ABC \), let \( \overrightarrow{AB}, \overrightarrow{BC} \) and \( \overrightarrow{CA} \) be denoted by \( u, v \) and \( w \). Show that

\[ u \times v = v \times w = w \times u, \]

and hence obtain the sine rule for \( \triangle ABC \).

(b) Given any three vectors \( p, q, r \) such that

\[ p \times q = q \times r = r \times p, \]

with \( |p \times q| \neq 0 \), show that

\[ p + q + r = 0. \]

10. (a) Using the identity \( a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \), deduce that

\[
\begin{align*}
(i) & \quad (a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c), \\
(ii) & \quad a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0.
\end{align*}
\]

Relate the case \( c = a, d = b \) of (i) to a well-known trigonometric identity.

Evaluate \( (a \times b) \times (c \times d) \) in two distinct ways and use the result to display explicitly a linear dependence relation amongst the four vectors \( a, b, c \) and \( d \).

(b) Given \( [a, b, c] = a \cdot (b \times c) \), show that

\[ [a \times b, b \times c, c \times a] = [a, b, c]^2. \]

*11. For \( \phi, \theta \in \mathbb{R} \), let the vectors \( e_x, e_\theta \) and \( e_\phi \) in \( \mathbb{R}^3 \) be defined in terms of the Cartesian basis \((i, j, k)\) by

\[
\begin{align*}
e_x &= \cos \phi \sin \theta i + \sin \phi \sin \theta j + \cos \theta k, \\
e_\theta &= \cos \phi \cos \theta i + \sin \phi \cos \theta j - \sin \theta k, \\
e_\phi &= -\sin \phi i + \cos \phi j.
\end{align*}
\]

Show that \((e_x, e_\theta, e_\phi)\) constitute an orthonormal right-handed basis. Discuss the significance of this [local] basis.

12. The set \( X \) contains the six real vectors

\[(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1).\]

Find two different subsets \( Y \) of \( X \) whose members are linearly independent, each of which yields a linearly dependent subset of \( X \) whenever any element \( x \in X \) with \( x \notin Y \) is adjoined to \( Y \).

13. Let \( V \) be the set of all vectors \( x = (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \) (\( n \geq 4 \)) such that their components satisfy

\[ x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n-3. \]

Find a basis for \( V \).

14. Let \( x \) and \( y \) be non-zero vectors in \( \mathbb{R}^n \) with scalar product denoted by \( x \cdot y \). Prove that

\[ (x \cdot y)^2 \leq (x \cdot x)(y \cdot y), \]

and prove also that \( (x \cdot y)^2 = (x \cdot x)(y \cdot y) \) if and only if \( x = \lambda y \) for some scalar \( \lambda \).

(a) By considering suitable vectors in \( \mathbb{R}^3 \), or otherwise, prove that the inequality

\[ x^2 + y^2 + z^2 \geq yz + zx + xy \]

holds for any real numbers \( x, y \) and \( z \).

(b) By considering suitable vectors in \( \mathbb{R}^4 \), or otherwise, show that only one choice of real numbers \( x, y \) and \( z \) satisfies

\[ 3(x^2 + y^2 + z^2 + 4) - 2(yz + zx + xy) - 4(x + y + z) = 0, \]

and find these numbers.