A1a **Vectors and Matrices: Example Sheet 1** Michaelmas 2017

A $*$ denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** *necessarily harder than unstarred questions.*

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1. Let S be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in S$ then
$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1.$$
Obtain the same result geometrically by considering the line containing the centre of the circle and the point 3.

2. Given $|z| = 1$ and $\arg z = \theta$, find both algebraically and geometrically the modulus-argument forms of
$$\text{(i)} \quad 1 + z, \qquad \text{(ii)} \quad 1 - z\,.$$
Show that the locus of w as z varies with $|z| = 1$, where w is given by
$$w^2 = \left(\frac{1-z}{1+z}\right)\,,$$
is a pair of straight lines.

3. Use complex numbers to show that the medians of a triangle are concurrent.
Hint: represent the vertices of the triangle by complex numbers z_1, z_2 and z_3 (or 0, z_1 and z_2 if you prefer), then write down equations for two of the medians and find their intersection.

$*$4. Express
$$I = \frac{z^5 - 1}{z - 1}$$
as a polynomial in z. By considering the complex fifth root of unity ω, obtain the four factors of I linear in z. Hence write I as the product of two real quadratic factors. By considering the term in z^2 in the identity so obtained for I, show that
$$4\cos\frac{\pi}{5}\sin\frac{\pi}{10} = 1\,.$$

5. Show the equation $\sin z = 2$ has infinitely many solutions.

6. (a) Let $z, a, b \in \mathbb{C}$ $(a \neq b)$ correspond to the points P, A, B of the Argand plane. Let C_λ be the locus of P defined by
$$PA/PB = \lambda\,,$$
where λ is a fixed real positive constant. Show that C_λ is a circle, unless $\lambda = 1$, and find its centre and radius. What if $\lambda = 1$?

   $*$(b) For the case $a = -b = p,\ p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that the curve
$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$
is a circle passing through A and B with its centre on the perpendicular bisector of AB.
Show that the circles C_λ and S_μ are orthogonal for all λ, μ.

7. Show by vector methods that the altitudes of a triangle are concurrent.
Hint: let the altitudes AD, BE of $\triangle ABC$ meet at H, and show that CH is perpendicular to AB.

8. Given that vectors $\mathbf{x}$ and $\mathbf{y}$ satisfy
$$\mathbf{x} + \mathbf{y}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{a}\,,$$
for fixed vector $\mathbf{a}$, show that
$$(\mathbf{x}\cdot\mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2 + |\mathbf{y}|^2}\,.$$

Deduce using the Schwarz inequality (or otherwise) that

$$|\mathbf{x}|(1+|\mathbf{y}|^2) \geqslant |\mathbf{a}| \geqslant |\mathbf{x}| \ .$$

Explain the circumstances under which either of the inequalities can be replaced by equalities, and describe the relation between $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{a}$ in these circumstances.

9. (a) In $\triangle ABC$, let $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ be denoted by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Show that

$$\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{w} = \mathbf{w}\times\mathbf{u}\,, \qquad (*)$$

and hence obtain the sine rule for $\triangle ABC$.

(b) Given *any* three vectors $\mathbf{p},\mathbf{q},\mathbf{r}$ such that

$$\mathbf{p}\times\mathbf{q} = \mathbf{q}\times\mathbf{r} = \mathbf{r}\times\mathbf{p}\,, \qquad (\dagger)$$

with $|\mathbf{p}\times\mathbf{q}| \neq 0$, show that

$$\mathbf{p}+\mathbf{q}+\mathbf{r} = \mathbf{0}\,.$$

10. (a) Using the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, deduce that

$$\begin{aligned} &\text{(i)} \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}),\\ &\text{(ii)} \quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}. \end{aligned}$$

Relate the case $\mathbf{c}=\mathbf{a}$, $\mathbf{d}=\mathbf{b}$ of (i) to a well-known trigonometric identity.
Evaluate $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ in two distinct ways and use the result to display explicitly a linear dependence relation amongst the four vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{d}$.

(b) Given $[\mathbf{a},\mathbf{b},\mathbf{c}] \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, show that

$$[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = [\mathbf{a},\mathbf{b},\mathbf{c}]^2.$$

∗11. For $\phi,\theta\in\mathbb{R}$, let the vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ in $\mathbb{R}^3$ be defined in terms of the Cartesian basis $(\mathbf{i},\mathbf{j},\mathbf{k})$ by

$$\begin{aligned} \mathbf{e}_r &= \cos\phi\sin\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k}\,,\\ \mathbf{e}_\theta &= \cos\phi\cos\theta\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j} - \sin\theta\,\mathbf{k}\,,\\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}\,. \end{aligned}$$

Show that $(\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\phi)$ constitute an orthonormal right-handed basis. Discuss the significance of this [local] basis.

12. The set X contains the six real vectors

$$(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1),\ (0,1,1,0),\ (0,1,0,1),\ (0,0,1,1).$$

Find two different subsets Y of X whose members are linearly independent, each of which yields a linearly dependent subset of X whenever any element $x\in X$ with $x\notin Y$ is adjoined to Y.

13. Let V be the set of all vectors $\mathbf{x}=(x_1,\ldots,x_n)$ in $\mathbb{R}^n$ $(n\geqslant 4)$ such that their components satisfy

$$x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0 \quad \text{for} \quad i = 1,2,\ldots,n-3\,.$$

Find a basis for V.

14. Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in $\mathbb{R}^n$ with scalar product denoted by $\mathbf{x}\cdot\mathbf{y}$. Prove that

$$(\mathbf{x}\cdot\mathbf{y})^2 \leqslant (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y})\,,$$

and prove also that $(\mathbf{x}\cdot\mathbf{y})^2 = (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y})$ if and only if $\mathbf{x}=\lambda\mathbf{y}$ for some scalar λ.

(a) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, prove that the inequality

$$x^2+y^2+z^2 \geqslant yz+zx+xy$$

holds for any real numbers x, y and z.

(b) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that only one choice of real numbers x, y and z satisfies

$$3(x^2+y^2+z^2+4) - 2(yz+zx+xy) - 4(x+y+z) = 0\,,$$

and find these numbers.