1. Let $D$ be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in D$ then
\[
\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1.
\]
Obtain the same result geometrically [start by considering the line through the centre of the circle and the point 3].

2. Given $|z| = 1$ and $\arg z = \theta$, find both algebraically and geometrically the modulus-argument forms of
\[
(i) \quad 1 + z, \quad (ii) \quad 1 - z.
\]
Show that the locus of $w$ as $z$ varies with $|z| = 1$, where $w$ is given by
\[
w^2 = \frac{1 - z}{1 + z},
\]
is a pair of straight lines.

3. Consider a triangle in the complex plane with vertices at 0, $z_1$ and $z_2$. Write down an expression for the general point on the median through $z_1$, and a similar expression for the general point on the median through $z_2$. Show that the three medians of the triangle are concurrent.

4. Express
\[
I = \frac{z^5 - 1}{z - 1}
\]
as a polynomial in $z$. By considering the complex fifth root of unity $\omega$, obtain the four factors of $I$ linear in $z$. Hence write $I$ as the product of two real quadratic factors. By considering the term in $z^2$ in the identity so obtained for $I$, show that
\[
4 \cos \frac{\pi}{5} \sin \frac{\pi}{10} = 1.
\]

5. Find all complex numbers $z$ that satisfy $\sin z = 2$.

6. (a) Let $z, a, b \in \mathbb{C}$ ($a \neq b$) correspond to points $P, A, B$ in the Argand diagram. Let $C_\lambda$ be the locus of $P$ defined by
\[
\frac{PA}{PB} = \lambda,
\]
where $\lambda$ is a fixed real positive constant. Show that $C_\lambda$ is a circle if $\lambda \neq 1$, and find its centre and radius. What happens if $\lambda = 1$?

(b) For the case $a = -b = p$, $p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that the curve
\[
S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}
\]
is a circle passing through $A$ and $B$ with its centre on the perpendicular bisector of $AB$.
Show that the circles $C_\lambda$ and $S_\mu$ intersect orthogonally for all $\lambda, \mu$.

7. Show by vector methods that the altitudes of a triangle are concurrent.
Hint: let the altitudes $AD, BE$ of $\triangle ABC$ meet at $H$, and show that $CH$ is perpendicular to $AB$.

8. Given that vectors $x$ and $y$ satisfy
\[
x + y(x \cdot y) = a,
\]
for a fixed vector $a$, show that
\[
(x \cdot y)^2 = \frac{|a|^2 - |x|^2}{2 + |y|^2}.
\]
Use an inequality involving $x \cdot y$ and the lengths of $x$ and $y$ to deduce that

$$|x|(1 + |y|^2) \geq |a| \geq |x|.$$ 

Explain the circumstances in which either of the inequalities above become equalities, and describe the relation between $x$, $y$ and $a$ in these circumstances.

9. (a) In $\triangle ABC$, let $\overrightarrow{AB} = u$, $\overrightarrow{BC} = v$ and $\overrightarrow{CA} = w$. Show that

$$u \times v = v \times w = w \times u,$$

and hence obtain the sine rule for $\triangle ABC$.

(b) Given any three vectors $p$, $q$, $r$ such that

$$p \times q = q \times r = r \times p,$$

and $|p \times q| \neq 0$, show that

$$p + q + r = 0.$$

10. Show that the line through the points $a$ and $b$ has equation

$$r = (1 - \lambda)a + \lambda b,$$

and that the plane through the points $a$, $b$ and $c$ has the equation

$$r = (1 - \mu - \nu)a + \mu b + \nu c,$$

where $\lambda, \mu$ and $\nu$ are scalars. Obtain forms of these equations that do not involve $\lambda, \mu, \nu$.

11. Let $a$, $b$, $c$, $d$ be fixed vectors in three dimensions. For each of the following equations, find all solutions for $r$:

(i) $r + r \times d = c$; (ii) $r + (r \cdot a)b = c$.

[In (ii), consider separately the cases $a \cdot b \neq -1$ and $a \cdot b = -1$.]

12. (a) Using the identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, show that

(i) $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$,

(ii) $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$.

Relate the case $c = a$, $d = b$ of (i) to a well-known trigonometric identity.

Evaluate $(a \times b) \times (c \times d)$ in two distinct ways and compare the results to find an explicit linear combination of the four vectors $a$, $b$, $c$, $d$ that is zero.

(b) Given $[a, b, c] \equiv a \cdot (b \times c)$, show that

$$[a \times b, b \times c, c \times a] = [a, b, c]^2.$$ 

13. The vectors $e_r$, $e_\theta$, $e_\phi$ are defined in terms of the standard basis vectors $i$, $j$, $k$ by

$$e_r = \cos \phi \sin \theta i + \sin \phi \sin \theta j + \cos \theta k,$$

$$e_\theta = \cos \phi \cos \theta i + \sin \phi \cos \theta j - \sin \theta k,$$

$$e_\phi = -\sin \phi i + \cos \phi j$$

where $\theta$ and $\phi$ are real. Show, as efficiently as possible, that $e_r$, $e_\theta$, $e_\phi$ are an orthonormal right-handed set.

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