

A1a

Vectors and Matrices: Example Sheet 1

Michaelmas 2011

A * denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** necessarily harder than unstarred questions.

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1. Use complex numbers to show that the medians of a triangle are concurrent.

Hint: represent the vertices of the triangle by complex numbers z_1, z_2 and z_3 (or $0, z_1$ and z_2 if you prefer), then write down equations for two of the medians and find their intersection.

2. Let S be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in S$ then

$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1.$$

Obtain the same result geometrically by considering the line containing the centre of the circle and the point 3.

3. Given $|z| = 1$ and $\arg z = \theta$, write in modulus argument form

$$(i) \quad 1 + z, \quad (ii) \quad 1 - z.$$

Show that w given by

$$w^2 = \left(\frac{1 - z}{1 + z} \right),$$

describes two fixed straight lines in the complex plane.

- *4. Express

$$I = \frac{z^5 - 1}{z - 1}$$

as a polynomial in z . By considering the complex fifth root of unity ω , obtain the four factors of I linear in z . Hence write I as the product of two real quadratic factors. By considering the term in z^2 in the identity so obtained for I , show that

$$4 \cos \frac{\pi}{5} \sin \frac{\pi}{10} = 1.$$

5. Explain (very briefly) why the equation $\sin z = 2$ has infinitely many solutions.
6. (a) Let $z, a, b \in \mathbb{C}$ ($a \neq b$) correspond to the points P, A, B of the Argand plane. Let C_λ be the locus of P defined by

$$PA/PB = \lambda,$$

where λ is a fixed real positive constant. Show that C_λ is a circle, unless $\lambda = 1$, and find its centre and radius. What if $\lambda = 1$?

- (b) For the case $a = -b = p$, $p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that the curve

$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$

is a circle passing through A and B with its centre on the perpendicular bisector of AB .

Show that the circles C_λ and S_μ are orthogonal for all λ, μ .

7. Show by vector methods that the altitudes of a triangle are concurrent.

Hint: let the altitudes AD, BE of $\triangle ABC$ meet at H , and show that CH is perpendicular to AB .

8. Given that vectors \mathbf{x} and \mathbf{y} satisfy

$$\mathbf{x} + \mathbf{y}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{a},$$

for fixed vector \mathbf{a} , show that

$$(\mathbf{x} \cdot \mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2 + |\mathbf{y}|^2}.$$

Deduce using the Schwarz inequality (or otherwise) that

$$|\mathbf{x}|(1 + |\mathbf{y}|^2) \geq |\mathbf{a}| \geq |\mathbf{x}|.$$

Explain the circumstances under which either of the inequalities can be replaced by equalities, and describe the relation between \mathbf{x} , \mathbf{y} and \mathbf{a} in these circumstances.

9. (a) In $\triangle ABC$, let \overrightarrow{AB} , \overrightarrow{BC} and \overrightarrow{CA} be denoted by \mathbf{u} , \mathbf{v} and \mathbf{w} . Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}, \quad (*)$$

and hence obtain the sine rule for $\triangle ABC$.

- (b) Given that $|\mathbf{u} \times \mathbf{v}| \neq 0$, show that (*) implies

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}.$$

10. (a) Using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, deduce that

$$(i) \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

$$(ii) \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$$

Relate the case $\mathbf{c} = \mathbf{a}$, $\mathbf{d} = \mathbf{b}$ of (i) to a well-known trigonometric identity.

Evaluate $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ in two distinct ways and use the result to display explicitly a linear dependence relation amongst the four vectors \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} .

- (b) Given $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, show that

$$[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.$$

- *11. For $\phi, \theta \in \mathbb{R}$, let the vectors \mathbf{e}_r , \mathbf{e}_θ and \mathbf{e}_ϕ in \mathbb{R}^3 be defined in terms of the Cartesian basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ by

$$\mathbf{e}_r = \cos \phi \sin \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \theta \mathbf{k},$$

$$\mathbf{e}_\theta = \cos \phi \cos \theta \mathbf{i} + \sin \phi \cos \theta \mathbf{j} - \sin \theta \mathbf{k},$$

$$\mathbf{e}_\phi = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}.$$

Show that $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ constitute an orthonormal right-handed basis. By reference to a textbook (or WIKIPEDIA if you must), or by inspiration, or otherwise, discuss the significance of this [local] basis.

12. The set X contains the six vectors

$$(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1).$$

Find two different subsets Y of X whose members are linearly independent, each of which yields a linearly dependent subset of X whenever any element $\in X$, $\notin Y$, is adjoined to Y .

13. Let V be the set of all vectors $x = (x_1, \dots, x_n)$ in \mathbb{R}^n ($n \geq 4$) such that

$$x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0 \quad \text{for } i = 1, 2, \dots, n-3.$$

Find a basis for V .

14. Let \mathbf{x} and \mathbf{y} be non-zero vectors in \mathbb{R}^n with scalar product denoted by $\mathbf{x} \cdot \mathbf{y}$. Prove that

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}),$$

and prove also that $(\mathbf{x} \cdot \mathbf{y})^2 = (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$ if and only if $\mathbf{x} = \lambda \mathbf{y}$ for some scalar λ .

- (a) By considering suitable vectors in \mathbb{R}^3 , or otherwise, prove that the inequality

$$x^2 + y^2 + z^2 \geq yz + zx + xy$$

holds for any real numbers x , y and z .

- (b) By considering suitable vectors in \mathbb{R}^4 , or otherwise, show that only one choice of real numbers x , y and z satisfies

$$3(x^2 + y^2 + z^2 + 4) - 2(yz + zx + xy) - 4(x + y + z) = 0,$$

and find these numbers.