

A1b **Vectors and Matrices: Example Sheet 2**

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1. In the following, the indices i, j, k, l take the values 1, 2, 3, and the summation convention applies. In particular, $n_i n_i = 1$; i.e., n_i are the components of a unit vector \mathbf{n} .

(a) Simplify the following expressions:

$$\delta_{ij} a_j, \quad \delta_{ij} \delta_{jk}, \quad \delta_{ij} \delta_{ji}, \quad \delta_{ij} n_i n_j, \quad \varepsilon_{ijk} \delta_{jk}, \quad \varepsilon_{ijk} \varepsilon_{ijl}, \quad \varepsilon_{ijk} \varepsilon_{ikj}, \quad \varepsilon_{ijk} (\mathbf{a} \times \mathbf{b})_k.$$

(b) Given that $A_{ij} = \varepsilon_{ijk} a_k$ (for all i, j), show that $2a_k = \varepsilon_{kij} A_{ij}$ (for all k).

(c) Show that $\varepsilon_{ijk} s_{ij} = 0$ (for all k) if and only if $s_{ij} = s_{ji}$ (for all i, j).

(d) Given that $N_{ij} = \delta_{ij} - \varepsilon_{ijk} n_k + n_i n_j$ and $M_{ij} = \delta_{ij} + \varepsilon_{ijk} n_k$, show that $N_{ij} M_{jk} = 2\delta_{ik}$.

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and \mathbf{d} be fixed vectors in \mathbb{R}^3 . In each of cases (i) and (ii) find all vectors \mathbf{r} such that

$$(i) \quad \mathbf{r} + \mathbf{r} \times \mathbf{d} = \mathbf{c}, \quad (ii) \quad \mathbf{r} + (\mathbf{r} \cdot \mathbf{a}) \mathbf{b} = \mathbf{c}.$$

In (ii) consider separately the $\mathbf{a} \cdot \mathbf{b} \neq -1$ and $\mathbf{a} \cdot \mathbf{b} = -1$ subcases.

Hint: given \mathbf{r}_0 solving (ii) for $\mathbf{a} \cdot \mathbf{b} = -1$, show that $\mathbf{r}_0 + \lambda \mathbf{b}$ is another solution for an arbitrary scalar λ .

3. In \mathbb{R}^3 show that the straight line through the points \mathbf{a} and \mathbf{b} has equation

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b},$$

and that the plane through the points \mathbf{a}, \mathbf{b} and \mathbf{c} has the equation

$$\mathbf{r} = (1 - \mu - \nu)\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c},$$

where λ, μ and ν are scalars. Obtain forms of these equations that do not involve λ, μ, ν .

4. (a) Let λ be a scalar, and let \mathbf{m}, \mathbf{u} and \mathbf{a} be fixed vectors in \mathbb{R}^3 such that $\mathbf{m} \cdot \mathbf{u} = 0$ and $\mathbf{a} \cdot \mathbf{u} \neq 0$. Show that the straight line $\mathbf{r} \times \mathbf{u} = \mathbf{m}$ meets the plane $\mathbf{r} \cdot \mathbf{a} = \lambda$ in the point

$$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{m} + \lambda \mathbf{u}}{\mathbf{a} \cdot \mathbf{u}}.$$

Explain in detail the geometrical meaning of the condition $\mathbf{a} \cdot \mathbf{u} \neq 0$.

- (b) In \mathbb{R}^3 show that if \mathbf{r} lies in the planes $\mathbf{r} \cdot \mathbf{a} = \lambda$ and $\mathbf{r} \cdot \mathbf{b} = \mu$, for fixed non-zero vectors \mathbf{a} and \mathbf{b} , and scalars λ and μ , show that

$$\mathbf{r} \times (\mathbf{a} \times \mathbf{b}) = \mu \mathbf{a} - \lambda \mathbf{b}. \quad (*)$$

Conversely, given $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$, show that (*) implies both $\mathbf{r} \cdot \mathbf{a} = \lambda$ and $\mathbf{r} \cdot \mathbf{b} = \mu$. Hence show that the intersection of two non-parallel planes is a line. Comment on the case in which $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

5. Let \mathbf{n} be a unit vector in \mathbb{R}^3 . Identify the image and kernel (null space) of each of the following linear maps $\mathbb{R}^3 \rightarrow \mathbb{R}^3$:

$$(a) \quad \mathcal{T} : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}) \mathbf{n}, \quad (b) \quad \mathcal{Q} : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{n} \times \mathbf{x}.$$

Show that $\mathcal{T}^2 = \mathcal{T}$ and interpret the map \mathcal{T} geometrically. Interpret the maps \mathcal{Q}^2 and $\mathcal{Q}^3 + \mathcal{Q}$, and show that $\mathcal{Q}^4 = \mathcal{T}$.

6. Give a geometrical description of the images and kernels of each of the linear maps of \mathbb{R}^3

$$(a) \quad (x, y, z) \mapsto (x + 2y + z, x + 2y + z, 2x + 4y + 2z),$$

$$(b) \quad (x, y, z) \mapsto (x + 2y + 3z, x - y + z, x + 5y + 5z).$$

7. A linear map $\mathcal{A} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is defined by $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{pmatrix}.$$

Find the kernel and image of \mathcal{A} for all real values of a and b .

8. A linear map $\mathcal{S} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} + \lambda(\mathbf{b} \cdot \mathbf{x}) \mathbf{a},$$

where λ is a scalar, and \mathbf{a} and \mathbf{b} are fixed, orthogonal unit vectors. By considering its effect on the vectors \mathbf{a} and \mathbf{b} , show that \mathcal{S} describes a shear in the direction of \mathbf{a} . Let $S(\lambda, \mathbf{a}, \mathbf{b})$ be the matrix with entries S_{ij} such that $x'_i = S_{ij}x_j$. Obtain an expression for S_{ij} in terms of the components of \mathbf{a} and \mathbf{b} and hence find the matrix $S(\lambda, \mathbf{a}, \mathbf{b})$. Evaluate its determinant[‡], and hence show that \mathcal{S} is an area-preserving map.

9. The linear map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \cos \theta \mathbf{x} + (\mathbf{x} \cdot \mathbf{n})(1 - \cos \theta) \mathbf{n} - \sin \theta (\mathbf{x} \times \mathbf{n}) \tag{†}$$

describes a rotation by angle θ in a positive sense about the unit vector \mathbf{n} . Verify this result in the case $\mathbf{n} = (0, 0, 1)$.

Show that (†) can be written in matrix form as

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{R}(\mathbf{n}, \theta) \mathbf{x}$$

where $\mathbf{R}(\mathbf{n}, \theta)$ is a matrix with entries R_{ij} which you should find explicitly in terms of $\delta_{ij}, \varepsilon_{ijk}$, etc. Hence show that

$$R_{ii} = 2 \cos \theta + 1, \quad \varepsilon_{ijk} R_{jk} = -2n_i \sin \theta.$$

Given that $\mathbf{R}(\mathbf{n}, \theta)$ is the matrix

$$\frac{1}{3} \begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix},$$

determine θ and \mathbf{n} .

10. Give examples of 2×2 real matrices representing the following transformations in \mathbb{R}^2 : (a) reflection, (b) dilatation (enlargement), (c) shear, and (d) rotation. Which of these types of transformation are always represented by a 2×2 matrix with determinant $+1$?

If maps \mathcal{A} and \mathcal{B} are both shears, will $\mathcal{A}\mathcal{B}$ be the same as $\mathcal{B}\mathcal{A}$ in general? Justify your answer.

11. Suppose that \mathbf{A} and \mathbf{B} are both Hermitian matrices. Show that $\mathbf{AB} + \mathbf{BA}$ is Hermitian. Also show that \mathbf{AB} is Hermitian if and only if \mathbf{A} and \mathbf{B} commute.

*12. Let $\mathbf{R}(\mathbf{n}, \theta)$ be the matrix defined by the linear map (†) of question 9, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the standard mutually orthogonal unit vectors in \mathbb{R}^3 .

(a) Show that the matrix $\mathbf{R}(\mathbf{i}, \frac{\pi}{2})\mathbf{R}(\mathbf{j}, \frac{\pi}{2})$ is orthogonal, has determinant one, and is not equal to the matrix $\mathbf{R}(\mathbf{j}, \frac{\pi}{2})\mathbf{R}(\mathbf{i}, \frac{\pi}{2})$.

(b) Reflection in a plane through the origin in \mathbb{R}^3 , with unit normal \mathbf{n} , is a linear map such that

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n}) \mathbf{n}.$$

In matrix notation $\mathbf{x}' = \mathbf{H}(\mathbf{n}) \mathbf{x}$ for matrix $\mathbf{H}(\mathbf{n})$. Show by geometrical and algebraic means that the map $\mathbf{x} \mapsto \mathbf{x}' = -\mathbf{H}(\mathbf{n})\mathbf{x}$, describes a rotation of angle π about \mathbf{n} .

(c) A vector \mathbf{x} has components (x, y, z) in a (Cartesian) coordinate system S . It has components (x', y', z') in a coordinate system S' obtained from S by anti-clockwise rotation through angle α about axis \mathbf{k} . Show, geometrically, that the components in coordinate system S' are related to those in S by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{R}(\mathbf{k}, -\alpha) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(d) Given that

$$\mathbf{n}_{\pm} = \cos \left(\frac{1}{2}\theta\right) \mathbf{i} \pm \sin \left(\frac{1}{2}\theta\right) \mathbf{j},$$

prove that

$$\mathbf{H}(\mathbf{i})\mathbf{H}(\mathbf{n}_{-}) = \mathbf{H}(\mathbf{n}_{+})\mathbf{H}(\mathbf{i}) = \mathbf{R}(\mathbf{k}, \theta),$$

and give diagrams to exhibit the geometrical meaning of this result.

[‡] You may need to return to this question if determinants have not been covered yet.