1. In the following, the indices $i,j,k,l$ take the values 1, 2, 3, and the summation convention applies. In particular, $n_in_i = 1$; i.e., $n_i$ are the components of a unit vector $\mathbf{n}$.
   
   (a) Simplify the following expressions:
   
   \[ \delta_{ij}a_i, \quad \delta_{ij}\delta_{kl}, \quad \delta_{ij}\delta_{jk}, \quad \varepsilon_{ijk}\delta_{jk}, \quad \varepsilon_{ijk}\varepsilon_{ijl}, \quad \varepsilon_{ijk}\varepsilon_{ikl}, \quad \varepsilon_{ij}(\mathbf{a} \times \mathbf{b})_k. \]

   (b) Given that $A_{ij} = \varepsilon_{ijk}a_k$ (for all $i,j$), show that $2a_k = \varepsilon_{kij}A_{ij}$ (for all $k$).
   
   (c) Show that $\varepsilon_{ijk}s_{ij} = 0$ (for all $k$) if and only if $s_{ij} = s_{ji}$ (for all $i,j$).
   
   (d) Given that $N_{ij} = \delta_{ij} - \varepsilon_{ijk}n_k + n_i n_j$ and $M_{ij} = \delta_{ij} + \varepsilon_{ijk}n_k$, show that $N_{ij}M_{jk} = 2\delta_{ik}$.

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ be fixed vectors in $\mathbb{R}^3$. In each case of (i) and (ii) find all vectors $\mathbf{r}$ such that

   (i) $\mathbf{r} + \mathbf{r} \times \mathbf{d} = \mathbf{c}$,  
   (ii) $\mathbf{r} + (\mathbf{r} \cdot \mathbf{a}) \mathbf{b} = \mathbf{c}$.

   In (ii) consider separately the $\mathbf{a} \cdot \mathbf{b} \neq -1$ and $\mathbf{a} \cdot \mathbf{b} = -1$ subcases.

   *Hint:* given $\mathbf{r}_0$ solving (ii) for $\mathbf{a} \cdot \mathbf{b} = -1$, show that $\mathbf{r}_0 + \lambda \mathbf{b}$ is another solution for an arbitrary scalar $\lambda$.

3. In $\mathbb{R}^3$ show that the straight line through the points $\mathbf{a}$ and $\mathbf{b}$ has equation

   \[ \mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda \mathbf{b}, \]

   and that the plane through the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ has the equation

   \[ \mathbf{r} = (1 - \mu - \nu)\mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}, \]

   where $\lambda, \mu$ and $\nu$ are scalars. Obtain forms of these equations that do not involve $\lambda, \mu, \nu$.

4. (a) Let $\lambda$ be a scalar, and let $\mathbf{m}, \mathbf{u}$ and $\mathbf{a}$ be fixed vectors in $\mathbb{R}^3$ such that $\mathbf{m} \cdot \mathbf{u} = 0$ and $\mathbf{a} \cdot \mathbf{u} \neq 0$.

   Show that the straight line $\mathbf{r} \times \mathbf{u} = \mathbf{m}$ meets the plane $\mathbf{r} \cdot \mathbf{a} = \lambda$ in the point

   \[ \mathbf{r} = \frac{\mathbf{a} \times \mathbf{m} + \lambda \mathbf{u}}{\mathbf{a} \cdot \mathbf{u}}. \]

   Explain in detail the geometrical meaning of the condition $\mathbf{a} \cdot \mathbf{u} \neq 0$.

   (b) In $\mathbb{R}^3$ show that if $\mathbf{r}$ lies in the planes $\mathbf{r} \cdot \mathbf{a} = \lambda$ and $\mathbf{r} \cdot \mathbf{b} = \mu$, for fixed non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, and scalars $\lambda$ and $\mu$, show that

   \[ \mathbf{r} \times (\mathbf{a} \times \mathbf{b}) = \mu \mathbf{a} - \lambda \mathbf{b}. \]  

   Conversely, given $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$, show that (a) implies both $\mathbf{r} \cdot \mathbf{a} = \lambda$ and $\mathbf{r} \cdot \mathbf{b} = \mu$. Hence deduce that the intersection of two non-parallel planes is a line. Comment on the case in which $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

5. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. Identify the image and kernel (null space) of each of the following linear maps $\mathbb{R}^3 \to \mathbb{R}^3$:

   (a) $T : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}) \mathbf{n}$,  
   (b) $Q : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{n} \times \mathbf{x}$.

   Show that $T^2 = T$ and interpret the map $T$ geometrically. Interpret the maps $Q^2$ and $Q^3 + Q$, and show that $Q^4 = T$.

6. Give a geometrical description of the images and kernels of each of the linear maps of $\mathbb{R}^3$

   (a) $(x, y, z) \mapsto (x + 2y + z, x + 2y + 2z, 2x + 4y + 2z)$,
   (b) $(x, y, z) \mapsto (x + 2y + 3z, x - y + z, x + 5y + 5z)$.

7. A linear map $A : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by $\mathbf{x} \mapsto A\mathbf{x}$ where

   \[ A = \begin{pmatrix} a & a & b & a \\ a & a & 0 & b \\ a & b & a & b \\ a & b & a & 0 \end{pmatrix}. \]

   Find the kernel and image of $A$ for all real values of $a$ and $b$. 

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**A1b Vectors and Matrices: Example Sheet 2**

**Corrections and suggestions should be emailed to N.Peake@damtp.cam.ac.uk, Michaelmas 2014.**

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**Mathematical Tripos IA: Vectors and Matrices**

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8. A linear map \( S : \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by
\[
x \mapsto x' = x + \lambda (b \cdot x) a,
\]
where \( \lambda \) is a scalar, and \( a \) and \( b \) are fixed, orthogonal unit vectors. By considering its effect on the vectors \( a \) and \( b \), show that \( S \) describes a shear in the direction of \( a \). Let \( S(\lambda, a, b) \) be the matrix with entries \( S_{ij} \) such that \( x'_i = S_{ij}x_j \). Obtain an expression for \( S_{ij} \) in terms of the components of \( a \) and \( b \) and hence find the matrix \( S(\lambda, a, b) \). Evaluate its determinant, and hence deduce that \( S \) is an area-preserving map.

9. The linear map \( \mathbb{R}^3 \to \mathbb{R}^3 \) defined by
\[
x \mapsto x' = \cos \theta x + (x \cdot n) (1 - \cos \theta) n - \sin \theta (x \times n)
\]
describes a rotation by angle \( \theta \) in a positive sense about the unit vector \( n \). Verify this by considering the case of \( n = (0, 0, 1) \).

Show that (†) can be written in matrix form as
\[
x \mapsto x' = R(n, \theta) x
\]
where \( R(n, \theta) \) is a matrix with entries \( R_{ij} \) which you should find explicitly in terms of \( \delta_{ij}, \varepsilon_{ijk} \), etc.

Hence show that
\[ R_{ii} = 2 \cos \theta + 1 \quad \varepsilon_{ijk} R_{jk} = -2n_i \sin \theta. \]

Given that \( R(n, \theta) \) is the matrix
\[
\frac{1}{3} \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix},
\]
determine \( \theta \) and \( n \).

10. Give examples of \( 2 \times 2 \) real matrices representing the following transformations in \( \mathbb{R}^2 \): (a) reflection, (b) dilatation (enlargement), (c) shear, and (d) rotation. Which of these types of transformation are always represented by a \( 2 \times 2 \) matrix with determinant +1?

If maps \( A \) and \( B \) are both shears, will \( AB \) be the same as \( BA \) in general? Justify your answer.

11. Suppose that \( A \) and \( B \) are both Hermitian matrices. Show that \( AB + BA \) is Hermitian. Also show that \( AB \) is Hermitian if and only if \( A \) and \( B \) commute.

12. Let \( R(n, \theta) \) be the matrix defined by the linear map (†) of question 9, and let \( i, j, k \) be the standard mutually orthogonal unit vectors in \( \mathbb{R}^3 \).

(a) Show that the matrix \( R(i, \frac{\pi}{2})R(j, \frac{\pi}{2}) \) is orthogonal, has determinant one, and is not equal to the matrix \( R(j, \frac{\pi}{2})R(i, \frac{\pi}{2}) \).

(b) Reflection in a plane through the origin in \( \mathbb{R}^3 \), with unit normal \( n \), is a linear map such that
\[
x \mapsto x' = x - 2(x \cdot n) n.
\]

In matrix notation \( x' = H(n)x \) for matrix \( H(n) \). Show by geometrical and algebraic means that the map \( x \mapsto x' = -H(n)x \), describes a rotation of angle \( \pi \) about \( n \).

(c) A vector \( x \) has components \((x, y, z)\) in a (Cartesian) coordinate system \( S \). It has components \((x', y', z')\) in a coordinate system \( S' \) obtained from \( S \) by anti-clockwise rotation through angle \( \alpha \) about axis \( k \). Show, geometrically, that the components in coordinate system \( S' \) are related to those in \( S \) by
\[
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R(k, -\alpha) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

(d) Given that
\[
n_k = \cos (\frac{1}{2} \theta) i \pm \sin (\frac{1}{2} \theta) j,
\]
prove that
\[
H(i)H(n_-) = H(n_+)H(i) = R(k, \theta),
\]
and give diagrams to exhibit the geometrical meaning of this result.

\[ \text{† You may need to return to this question if determinants have not been covered yet.} \]