A1b **Vectors and Matrices: Example Sheet 2** Michaelmas 2011

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1. In the following, the indices i, j, k, l take the values $1, 2, 3$, and the summation convention applies. In particular, $n_i n_i = 1$; i.e., n_i are the components of a unit vector $\mathbf{n}$.

   (a) Simplify the following expressions:
   $$\delta_{ij}a_j\,, \quad \delta_{ij}\delta_{jk}\,, \quad \delta_{ij}\delta_{ji}\,, \quad \delta_{ij}n_i n_j\,, \qquad \varepsilon_{ijk}\delta_{jk}\,, \qquad \varepsilon_{ijk}\varepsilon_{ijl}\,, \quad \varepsilon_{ijk}\varepsilon_{ikj}\,, \quad \varepsilon_{ijk}(\mathbf{a}\times\mathbf{b})_k.$$

   (b) Given that $A_{ij} = \varepsilon_{ijk}a_k$ (for all i, j), show that $2a_k = \varepsilon_{kij}A_{ij}$ (for all k).

   (c) Show that $\varepsilon_{ijk}s_{ij} = 0$ (for all k) if and only if $s_{ij} = s_{ji}$ (for all i, j).

   (d) Given that $N_{ij} = \delta_{ij} - \varepsilon_{ijk}n_k + n_i n_j$ and $M_{ij} = \delta_{ij} + \varepsilon_{ijk}n_k$, show that $N_{ij}M_{jk} = 2\delta_{ik}$.

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ be fixed vectors in $\mathbb{R}^3$. In each of cases (i) and (ii) find all vectors $\mathbf{r}$ such that
   $$\text{(i)}\;\; \mathbf{r} + \mathbf{r}\times\mathbf{d} = \mathbf{c}\,, \qquad \text{(ii)}\;\; \mathbf{r} + (\mathbf{r}\cdot\mathbf{a})\,\mathbf{b} = \mathbf{c}\,.$$
   In (ii) consider separately the $\mathbf{a}\cdot\mathbf{b} \neq -1$ and $\mathbf{a}\cdot\mathbf{b} = -1$ subcases.

   Hint: given $\mathbf{r}_0$ solving (ii) for $\mathbf{a}\cdot\mathbf{b} = -1$, show that $\mathbf{r}_0 + \lambda\mathbf{b}$ is another solution for an arbitrary scalar λ.

3. In $\mathbb{R}^3$ show that the straight line through the points $\mathbf{a}$ and $\mathbf{b}$ has equation
   $$\mathbf{r} \;=\; (1-\lambda)\mathbf{a} \;+\; \lambda\mathbf{b},$$
   and that the plane through the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ has the equation
   $$\mathbf{r} \;=\; (1-\mu-\nu)\mathbf{a} \;+\; \mu\mathbf{b} \;+\; \nu\mathbf{c},$$
   where λ, μ and ν are scalars. Obtain forms of these equations that do not involve λ, μ, ν.

4. (a) Let λ be a scalar, and let $\mathbf{m}, \mathbf{u}$ and $\mathbf{a}$ be fixed vectors in $\mathbb{R}^3$ such that $\mathbf{m}\cdot\mathbf{u} = 0$ and $\mathbf{a}\cdot\mathbf{u} \neq 0$. Show that the straight line $\mathbf{r}\times\mathbf{u} = \mathbf{m}$ meets the plane $\mathbf{r}\cdot\mathbf{a} = \lambda$ in the point
   $$\mathbf{r} \;=\; \frac{\mathbf{a}\times\mathbf{m} + \lambda\mathbf{u}}{\mathbf{a}\cdot\mathbf{u}}\,.$$
   Explain in detail the geometrical meaning of the condition $\mathbf{a}\cdot\mathbf{u} \neq 0$?

   (b) In $\mathbb{R}^3$ show that if $\mathbf{r}$ lies in the planes $\mathbf{r}\cdot\mathbf{a} = \lambda$ and $\mathbf{r}\cdot\mathbf{b} = \mu$, for fixed non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, and scalars λ and μ, show that
   $$\mathbf{r}\times(\mathbf{a}\times\mathbf{b}) \;=\; \mu\mathbf{a} - \lambda\mathbf{b}\,. \qquad (*)$$
   Conversely, given $\mathbf{a}\times\mathbf{b} \neq \mathbf{0}$, show that $(*)$ implies both $\mathbf{r}\cdot\mathbf{a} = \lambda$ and $\mathbf{r}\cdot\mathbf{b} = \mu$. Hence deduce that the intersection of two non-parallel planes is a line. Comment on the case in which $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

5. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. Identify the image and kernel (null space) of each of the following linear maps $\mathbb{R}^3 \to \mathbb{R}^3$:
   $$\text{(a)}\; \mathcal{T} : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\,\mathbf{n}\,, \qquad \text{(b)}\; \mathcal{Q} : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{n}\times\mathbf{x}\,.$$
   Show that $\mathcal{T}^2 = \mathcal{T}$ and interpret the map $\mathcal{T}$ geometrically. Interpret the maps $\mathcal{Q}^2$ and $\mathcal{Q}^3 + \mathcal{Q}$, and show that $\mathcal{Q}^4 = \mathcal{T}$.

6. Give a geometrical description of the images and kernels of each of the linear maps of $\mathbb{R}^3$

   (a) $(x, y, z) \mapsto (x + 2y + z, x + 2y + z, 2x + 4y + 2z)$,

   (b) $(x, y, z) \mapsto (x + 2y + 3z, x - y + z, x + 5y + 5z)$.

7. A linear map $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ where
   $$\mathsf{A} \;=\; \begin{pmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{pmatrix}.$$
   Find the kernel and image of $\mathcal{A}$ for all real values of a and b.

8. A linear map $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} + \lambda(\mathbf{b} \cdot \mathbf{x})\,\mathbf{a}\,,$$

where λ is a scalar, and $\mathbf{a}$ and $\mathbf{b}$ are fixed, orthogonal unit vectors. By considering its effect on the vectors $\mathbf{a}$ and $\mathbf{b}$, show that $\mathcal{S}$ describes a simple shear in the direction of $\mathbf{a}$. Let $\mathsf{S}(\lambda, \mathbf{a}, \mathbf{b})$ be the matrix with entries S_{ij} such that $x'_i = S_{ij} x_j$. Obtain an expression for S_{ij} in terms of the components of $\mathbf{a}$ and $\mathbf{b}$ and hence find the matrix $\mathsf{S}(\lambda, \mathbf{a}, \mathbf{b})$. Evaluate its determinant‡, and hence deduce that $\mathcal{S}$ is an area-preserving map.

9. The linear map $\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \cos\theta\,\mathbf{x} + (\mathbf{x} \cdot \mathbf{n})\,(1 - \cos\theta)\;\mathbf{n} - \sin\theta\,(\mathbf{x} \times \mathbf{n}) \qquad (\dagger)$$

describes a rotation by angle θ in a positive sense about the unit vector $\mathbf{n}$. Verify this by considering the case of $\mathbf{n} = (0, 0, 1)$.

Show that (†) can be written in matrix form as

$$\mathbf{x} \mapsto \mathbf{x}' = \mathsf{R}(\mathbf{n}, \theta)\,\mathbf{x}$$

where $\mathsf{R}(\mathbf{n}, \theta)$ is a matrix with entries R_{ij} which you should find explicitly in terms of $\delta_{ij}, \varepsilon_{ijk}$, etc. Hence show that

$$R_{ii} = 2\cos\theta + 1 \quad , \quad \varepsilon_{ijk} R_{jk} = -2n_i \sin\theta.$$

Given that $\mathsf{R}(\mathbf{n}, \theta)$ is the matrix

$$\frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix},$$

determine θ and $\mathbf{n}$.

10. Give examples of 2×2 real matrices representing the following transformations in $\mathbb{R}^2$: (a) reflection, (b) dilatation, (c) shear, and (d) rotation. Which of these types of transformation are always represented by a 2×2 matrix with determinant $+1$?

If maps $\mathcal{A}$ and $\mathcal{B}$ are both simple shears, will $\mathcal{AB}$ be the same as $\mathcal{BA}$ in general? Justify your answer.

11. Suppose that A and B are both Hermitian matrices. Show that $\mathsf{AB} + \mathsf{BA}$ is Hermitian. Also show that AB is Hermitian if and only if A and B commute.

*12. Let $\mathsf{R}(\mathbf{n}, \theta)$ be the matrix defined by the linear map (†) of question 9, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the standard mutually orthogonal unit vectors in $\mathbb{R}^3$.

(a) Show that the matrix $\mathsf{R}(\mathbf{i}, \frac{\pi}{2})\mathsf{R}(\mathbf{j}, \frac{\pi}{2})$ is orthogonal, has determinant one, and is not equal to the matrix $\mathsf{R}(\mathbf{j}, \frac{\pi}{2})\mathsf{R}(\mathbf{i}, \frac{\pi}{2})$.

(b) Reflection in a plane through the origin in $\mathbb{R}^3$, with unit normal $\mathbf{n}$, is a linear map such that

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}.$$

In matrix notation $\mathbf{x}' = \mathsf{H}(\mathbf{n})\,\mathbf{x}$ for matrix $\mathsf{H}(\mathbf{n})$. Show by geometrical and algebraic means that the map $\mathbf{x} \mapsto \mathbf{x}' = -\mathsf{H}(\mathbf{n})\mathbf{x}$, describes a rotation of angle π about $\mathbf{n}$.

(c) A vector $\mathbf{x}$ has components (x, y, z) in a (Cartesian) coordinate system S. It has components (x', y', z') in a coordinate system S' obtained from S by anti-clockwise rotation through angle α about axis $\mathbf{k}$. Show, geometrically, that the components in coordinate system S' are related to those in S by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathsf{R}(\mathbf{k}, -\alpha)\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(d) Given that

$$\mathbf{n}_\pm = \cos\left(\tfrac{1}{2}\theta\right)\,\mathbf{i} \;\pm\; \sin\left(\tfrac{1}{2}\theta\right)\,\mathbf{j}\,,$$

prove that

$$\mathsf{H}\,(\mathbf{i})\,\mathsf{H}\,(\mathbf{n}_-) = \mathsf{H}\,(\mathbf{n}_+)\,\mathsf{H}\,(\mathbf{i}) = \mathsf{R}\,(\mathbf{k}, \theta)\,,$$

and give diagrams to exhibit the geometrical meaning of this result.

‡ You may need to return to this question if determinants have not been covered yet.