Example Sheet 2

1. In the following, the indices $i, j, k, \ell$ take the values 1, 2, 3 and the summation convention applies.
   (a) Simplify the following expressions:
   
   \[
   \delta_{ij}v_j, \quad \delta_{ij}\delta_{jk}, \quad \delta_{ij}\delta_{ji}, \quad \varepsilon_{ijk}\delta_{jk}, \quad \varepsilon_{ijk}v_jv_k, \quad \varepsilon_{ijk}\varepsilon_{ij\ell}, \quad \varepsilon_{ijk}\varepsilon_{ikj}.
   \]

   (b) Given that $A_{ij} = \varepsilon_{ijk}a_k$ (for all $i, j$), show that $2a_k = \varepsilon_{kij}A_{ij}$ (for all $k$).

   (c) Show that $\varepsilon_{ijk}S_{ij} = 0$ (for all $k$) if and only if $S_{ij} = S_{ji}$ (for all $i, j$).

2. For vectors in $\mathbb{R}^3$, simplify $\varepsilon_{ijk} (a \times b)_k$ and deduce a standard formula for $c \times (a \times b)$.
   (a) Let $m, u$ and $a$ be fixed vectors in $\mathbb{R}^3$ such that $m \cdot u = 0$ and $a \cdot u \neq 0$. Show that the line $r = \frac{a \times m + \kappa u}{a \cdot u}$ meets the plane $r \cdot a = \kappa$ (a constant) in the point $r = \frac{a \times m + \kappa u}{a \cdot u}$.

   Explain clearly the geometrical meaning of the condition $a \cdot u \neq 0$.

   (b) Let $a$ and $b$ be vectors in $\mathbb{R}^3$ with $a \times b \neq 0$. Show that the planes $r \cdot a = \kappa$ and $r \cdot b = \rho$ (where $\kappa, \rho$ are constants) intersect in the line $r \times (a \times b) = \rho a - \kappa b$,

   i.e., show that every point that lies on both planes lies on the line and, conversely, every point on the line lies on both planes. What happens if $a \times b = 0$?

3. Show that $M_{ij} = \delta_{ij} + \varepsilon_{ijk}n_k$ and $N_{ij} = \delta_{ij} - \varepsilon_{ijk}n_k + n_in_j$ obey $N_{ij}M_{jk} = 2\delta_{ik}$, if $n_in_i = 1$ (indices take values 1, 2, 3 and the summation convention applies). Verify that $y = x + x \times n \iff y_i = M_{ij}x_j$,

   where $x, y, n$ are vectors in $\mathbb{R}^3$ with components $x_i, y_i, n_i$. Use these results to find $x$ in terms of $y$, given that $n$ is a unit vector.

4. The set $X$ consists of six vectors in $\mathbb{R}^4$:

   \[
   (1, 1, 0, 0), \quad (1, 0, 1, 0), \quad (1, 0, 0, 1), \quad (0, 1, 1, 0), \quad (0, 1, 0, 1), \quad (0, 0, 1, 1).
   \]

   Find two different subsets $Y$ of $X$ whose members are linearly independent, each of which yields a linearly dependent subset of $X$ whenever any element $v \in X$ with $v \notin Y$ is adjoined to $Y$.

5. Let $V$ be the set of all vectors $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ ($n \geq 4$) such that their components satisfy

   \[x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n - 3.\]

   Show that $V$ is a subspace of $\mathbb{R}^n$, and find a basis for $V$.

6. State the Cauchy-Schwarz inequality for vectors $u$ and $v$ in $\mathbb{R}^n$ and give a necessary and sufficient condition for equality to hold.
   (a) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, show that

   \[
x^2 + y^2 + z^2 \geq yz + zx + xy, \quad \text{for any real numbers} \quad x, y, z.
   \]

   (b) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that

   \[
   3(x^2 + y^2 + z^2 + 4) - 2(yz + zx + xy) - 4(x + y + z) = 0
   \]

   holds for unique real values of $x, y, z$, to be determined.
7. Let \( \mathbf{n} \) be a unit vector in \( \mathbb{R}^3 \). Identify the image and kernel (null space) of each of the following linear maps \( \mathbb{R}^3 \to \mathbb{R}^3 \):
   
   (a) \( T : \mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}) \mathbf{n} \),
   
   Show that \( T^2 = T \) and interpret the map \( T \) geometrically. Interpret the maps \( Q^2 \) and \( Q^3 + Q \), and show that \( Q^2 = T \).

8. Give a geometrical description of the images and kernels of each of the following linear maps on \( \mathbb{R}^3 \):
   
   (a) \( T : (x, y, z) \mapsto (x + 2y + z, x + 2y + z, 2x + 4y + 2z) \),
   
   (b) \( S : (x, y, z) \mapsto (x + 2y + 3z, x - y + z, x + 5y + 5z) \).

9. A linear map \( \mathbb{R}^4 \to \mathbb{R}^4 \) is defined by \( \mathbf{x} \mapsto M \mathbf{x} \) where
   
   \[
   M = \begin{pmatrix}
   a & a & b & a \\
   a & a & b & 0 \\
   a & b & a & b \\
   a & b & a & 0
   \end{pmatrix}.
   
   Find the image and kernel of this map for all real values of \( a \) and \( b \).

10. The linear map \( \mathbb{R}^3 \to \mathbb{R}^3 \) defined by
    
    \[
    \mathbf{x} \mapsto \mathbf{x}' = \cos \theta \mathbf{x} + (\mathbf{x} \cdot \mathbf{n}) (1 - \cos \theta) \mathbf{n} - \sin \theta (\mathbf{x} \times \mathbf{n})
    \]
    
    is a rotation by angle \( \theta \) in a positive sense about the unit vector \( \mathbf{n} \). Check this in the case \( \mathbf{n} = (0, 0, 1) \).
    
    Show that the expression given above for a general rotation can be written \( \mathbf{x}' = R \mathbf{x} \), where \( R \) is a matrix with entries \( R_{ij} \) that should be found explicitly (in terms of \( \theta, \mathbf{n}_1, \delta_{ij}, \varepsilon_{ijk} \)). Hence show that
    
    \[ R_{ii} = 2 \cos \theta + 1, \quad \varepsilon_{ijk} R_{jk} = -2 n_i \sin \theta. \]
    
    Determine \( \theta \) and \( \mathbf{n} \) for the rotation given by the matrix
    
    \[
    R = \frac{1}{3} \begin{pmatrix}
    2 & -1 & 2 \\
    2 & 2 & -1 \\
    -1 & 2 & 2
    \end{pmatrix}.
    \]

11. (a) Give examples of \( 2 \times 2 \) real matrices representing the following types of transformations in \( \mathbb{R}^2 \):
    
    (i) reflection; (ii) dilation (or scaling); (iii) shear; and (iv) rotation.
    
    Which of these types of transformation are always represented by a \( 2 \times 2 \) matrix with determinant +1? For which types (i)-(iv) do transformations \( A \) and \( B \) of the same type obey \( AB = BA \) in general?
    
    (b) A linear map \( \mathbb{R}^2 \to \mathbb{R}^2 \) with \( \mathbf{x} \mapsto \mathbf{x}' = M \mathbf{x} \) is defined by \( \mathbf{z} \mapsto \mathbf{z}' = c \mathbf{z} \) where \( \mathbf{z} = x_1 + i x_2 \), \( \mathbf{z}' = x'_1 + i x'_2 \) and \( c = a + i b \) is a fixed complex number. Find the real \( 2 \times 2 \) matrix \( M \) in terms of \( a \) and \( b \).
    
    Which types of transformations (i)-(iv) can be obtained for particular choices of \( c = a + i b \)?

12. Let \( R(\mathbf{n}, \theta) \) be the matrix corresponding to a rotation with angle \( \theta \) and axis \( \mathbf{n} \), as given in (i) of question 10. Let \( H(\mathbf{n}) \) be the matrix corresponding to reflection in a plane through the origin with unit normal \( \mathbf{n} \), as defined by
    
    \[
    \mathbf{x} \mapsto \mathbf{x}' = H(\mathbf{n}) \mathbf{x} = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n}) \mathbf{n}.
    \]
    
    In the following, \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are the standard orthonormal basis vectors in \( \mathbb{R}^3 \).
    
    (a) Find explicitly the matrices \( R(\mathbf{i}, \frac{\pi}{2}) \) and \( R(\mathbf{j}, \frac{\pi}{2}) \) and check that \( R(\mathbf{i}, \frac{\pi}{2}) R(\mathbf{j}, \frac{\pi}{2}) \neq R(\mathbf{j}, \frac{\pi}{2}) R(\mathbf{i}, \frac{\pi}{2}) \).
    
    (b) Show by both algebraic and geometrical means that the map \( \mathbf{x} \mapsto -H(\mathbf{n}) \mathbf{x} \) is a rotation through an angle \( \pi \) about \( \mathbf{n} \).
    
    (c) Given that \( \mathbf{n}_\pm = \cos \left( \frac{\theta}{2} \right) \mathbf{i} \pm \sin \left( \frac{\theta}{2} \right) \mathbf{j} \), prove that
    
    \[
    H(\mathbf{i}) H(\mathbf{n}_-) = H(\mathbf{n}_+) H(\mathbf{i}) = R(\mathbf{k}, \theta),
    \]
    
    and draw diagrams to explain the geometrical meaning of this result.

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