1. A square matrix $A$ is upper triangular if $A_{ij} = 0$ for $i > j$. Show that the eigenvalues of such a matrix are its diagonal entries: $\lambda_i = A_{ii}$ (no sum over $i$).

2. Show that the matrix

$$
M = \begin{pmatrix}
0 & 1 & 0 \\
-4 & 4 & 0 \\
-2 & 1 & 2
\end{pmatrix}
$$

has characteristic equation $(t - 2)^3 = 0$. Explain, as simply as possible, why $M$ is not diagonalisable.

3. Find $a$, $b$ and $c$ such that

$$
\begin{pmatrix}
1/3 & 0 & a \\
2/3 & 1/\sqrt{2} & b \\
2/3 & -1/\sqrt{2} & c
\end{pmatrix}
$$

is an orthogonal matrix. Does this condition determine $a$, $b$ and $c$ uniquely?

4. Determine the eigenvalues and eigenvectors of the symmetric matrix

$$
A = \begin{pmatrix}
3 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 2
\end{pmatrix}
$$

Use an identity of the form $P^TAP = D$, where $D$ is a diagonal matrix, to find $A^{-1}$.

5. Diagonalise the quadratic form in $\mathbb{R}^2$ defined by

$$
F(x, y) = (a \cos^2 \theta + b \sin^2 \theta) x^2 + 2(a - b)(\sin \theta \cos \theta) xy + (a \sin^2 \theta + b \cos^2 \theta) y^2,
$$

i.e., find its eigenvalues and principal axes ($a$, $b$ and $\theta$ are constants).

6. (i) A matrix $A$ is anti-hermitian, $A^\dagger = -A$; show that the eigenvalues of $A$ are pure-imaginary.
(ii) A matrix $U$ is unitary, $U^\dagger U = I$; show that the eigenvalues of $U$ have unit modulus.
(iii) In each of the cases (i) and (ii), show that eigenvectors with distinct eigenvalues are orthogonal.

7. Check, by direct calculation, that the Cayley-Hamilton Theorem holds for a general $2 \times 2$ matrix. Find the characteristic polynomial for

$$
A = \begin{pmatrix}
3 & 4 \\
-1 & -1
\end{pmatrix}
$$

and deduce that $A^2 = 2A - I$. Is $A$ diagonalisable?

Show by induction that

$$
A^n = \alpha_n A + \beta_n I, \quad n \geq 0,
$$

for real numbers $\alpha_n$ and $\beta_n$. Solve the recurrence relations (difference equations) satisfied by $\alpha_n$ and $\beta_n$ and hence find $A^n$ explicitly.

8. Define the $m \times n$ matrix $A$ that represents a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ with respect to general bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$.

(a) Taking $n = 2$, $m = 3$, let $T$ be defined by

$$
T : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad T : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}.
$$

Find the matrix $A$ with respect to the bases

$$
e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}; \quad f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
$$
(b) Taking \( n = m = 3 \), let \( T \) be reflection in the plane \( x_1 \sin \theta = x_2 \cos \theta \). Find the matrix \( A \) with respect to a convenient choice of bases (to be specified) such that \( e_i = f_i \ (i = 1, 2, 3) \).

(c) Taking \( n = m = 2 \), let \( T \) be the shear with parameter \( \lambda \) defined by

\[
T: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \lambda \\ 1 \end{pmatrix}.
\]

Find the matrix \( A \) when \( e_1 = f_1 \), and \( e_2 = f_2 \) are the standard basis vectors for \( \mathbb{R}^2 \); find also the matrix \( A' \) with respect to a new basis \( e_1' = f_1' = -e_2 \) and \( e_2' = f_2' = e_1 \). Show that \( A' = R^{-1}AR \) for a rotation matrix \( R \) and comment on this result.

9. The linear map \( S: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is defined in terms of its matrix \( A \) with respect to the standard basis by

\[
S: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + 9y \\ -4x + 7y \end{pmatrix}.
\]

Find the matrix \( B \) for \( S \) with respect to the basis \( \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \).

Show that \( B^n - I = n(B - I) \) for all positive integers \( n \), and hence determine \( A^n \).

10. Find all eigenvalues, and an orthonormal set of eigenvectors, of the matrices

\[
A = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.
\]

Hence sketch the surfaces

\[
5x^2 + 3y^2 + 3z^2 + 2\sqrt{3}xz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 - xy - yz - zx = 1.
\]

11. Let \( \Sigma \) be the surface in \( \mathbb{R}^3 \) given by

\[
2x^2 + 2xy + 4yz + z^2 = 1.
\]

By considering a suitable real symmetric matrix, show that there is a new orthonormal basis with associated coordinates \( u, v, w \) such that \( \Sigma \) is given by

\[
\lambda u^2 + \mu v^2 + \nu w^2 = 1,
\]

for constants \( \lambda, \mu, \nu \), to be determined. Find the minimum distance from a point on \( \Sigma \) to the origin. [You need not find the new basis vectors explicitly.]

12. If \( S \) is a real symmetric matrix and \( A \) is a real antisymmetric matrix, show that \( A + iS \) is anti-hermitian (see question 6, part (i), above) and deduce that

\[
\det(A + iS - I) \neq 0.
\]

Show that the matrix

\[
U = (I + A + iS)(I - A - iS)^{-1}
\]

is unitary. Find \( U \) when

\[
S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

and show that it has eigenvalues \( \pm(1 - i)/\sqrt{2} \).

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