A1d  Vectors and Matrices: Example Sheet 4  Michaelmas 2017

A * denotes a question, or part of a question, that should not be done at the expense of questions later on the sheet. Starred questions are not necessarily harder than unstarred questions.

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1. A matrix $A$ is said to be upper triangular if $A_{ij} = 0$ if $i > j$, i.e. if

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
0 & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{nn}
\end{pmatrix}.
\]

Show that the eigenvalues are $\lambda_i = A_{ii}$ ($i = 1, \ldots, n$, and obviously no sum).

2. Let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $A$ be a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. Explain how to represent $A$ by a matrix relative to the given bases.

(a) Taking $m = 2$, $n = 3$ and $A$ as the linear mapping for which

\[
A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix},
\]

where components are with respect to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, find the matrix of $A$ with respect to the bases

\[
e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}; \quad f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

(b) The mapping $A$ of $\mathbb{R}^3$ to itself is a reflection in the plane $x_1 \sin \theta = x_2 \cos \theta$. Find the matrix $A$ of $A$ with respect to any basis of your choice, which should be specified.

3. The linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + 9y \\ -4x + 7y \end{pmatrix}.
\]

Find the matrix $B$ of the map $A$ relative to the basis

\[
\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\},
\]

and interpret the map geometrically. Hence show that, for each positive integer $n$,

\[
B^n - I = n(B - I),
\]

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence evaluate $A^n$. Verify that $\det(A^n) = (\det A)^n$.

*4. Show that similar matrices have the same rank.

5. Show that the matrix

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}
\]

has characteristic equation $(t - 2)^3 = 0$. Explain (without doing any further calculations) why $A$ is not diagonalisable.
6. (a) Find \( a, b \) and \( c \) such that the matrix
\[
\begin{pmatrix}
\frac{1}{3} & 0 & a \\
\frac{2}{3} & \frac{1}{\sqrt{2}} & b \\
\frac{2}{3} & -\frac{1}{\sqrt{2}} & c
\end{pmatrix}
\]
is orthogonal. Does this condition determine \( a, b \) and \( c \) uniquely?

(b) Let
\[
A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]
Do you expect \( PAP^{-1} \) to be symmetric? Compute \( PAP^{-1} \). Were you right?

7. (a) Show that the Cayley-Hamilton theorem is true for all \( 2 \times 2 \) matrices.

(b) Let
\[
A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.
\]
Find the characteristic equation for \( A \). Verify that \( A^2 = 2A - I \). Is \( A \) diagonalisable?

Show by induction that \( A^n \) lies in the two-dimensional subspace (of the space of \( 2 \times 2 \) real matrices) spanned by \( A \) and \( I \), i.e. show that there exists real numbers \( \alpha_n \) and \( \beta_n \) such that
\[
A^n = \alpha_n A + \beta_n I.
\]
Find a recurrence relation (i.e. a difference equation) for \( \alpha_n \) and \( \beta_n \), and hence find an explicit formula for \( A^n \).

8. Determine the eigenvalues and eigenvectors of the symmetric matrix
\[
A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.
\]
Use an identity of the form \( P^TAP = D \), where \( D \) is a diagonal matrix, to find \( A^{-1} \).

9. Show that the eigenvalues of a unitary matrix have unit modulus. Show that if a unitary matrix has distinct eigenvalues then the eigenvectors are orthogonal.

10. A skew-Hermitian matrix, \( W \), is one such that \( W^\dagger = -W \). What can be said about the eigenvalues of a skew-Hermitian matrix? (Hint: consider \( H = iW \))?

If \( S \) is real symmetric and \( T \) is real skew-symmetric, show that \( T \pm iS \) is skew-Hermitian. State a property of the eigenvalues of \( T + iS \) and hence, or otherwise, show that
\[
det(T + iS - I) \neq 0.
\]
Show that the matrix
\[
U = (1 + T + iS)(1 - T - iS)^{-1}
\]
is unitary. For
\[
S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
show that the eigenvalues of \( U \) are \( \pm(1 - i)/\sqrt{2} \).

11. This is a continuation of question 8 on Example Sheet 2.

As in question 8 on Example Sheet 2 consider the linear map \( S : \mathbb{R}^2 \to \mathbb{R}^2 \)
\[
x \mapsto x' = x + \lambda(b \cdot x) a
\]
where \( \lambda \) is a real scalar, \( a \) and \( b \) are fixed orthogonal unit vectors. Let \( S(\lambda, a, b) \) be the matrix with elements \( S_{ij} \) such that \( x'_i = S_{ij}x_j \). Give diagrams illustrating the shears
\[
S_1 = S(\lambda, 1, 1), \quad S_2 = S(\lambda, 1, -1).
\]
Show that $S_1$ and $S_2$ are related by a similarity transformation

\[ S_2 = R^{-1} S_1 R, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Now let $S$ be the map defined by $(*)$ but from $\mathbb{R}^3$ to $\mathbb{R}^3$, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be unit vectors along the three perpendicular axes. Find the matrix $S$ in each of the cases

(i) $\mathbf{a} = \mathbf{i}, \mathbf{b} = \mathbf{j}$,
(ii) $\mathbf{a} = \mathbf{j}, \mathbf{b} = \mathbf{k}$,
(iii) $\mathbf{a} = \mathbf{k}, \mathbf{b} = \mathbf{i}$,

and interpret the corresponding simple shears. Show that any matrix of the form

\[
\begin{pmatrix}
1 & \lambda & \mu \\
0 & 1 & \nu \\
0 & 0 & 1
\end{pmatrix}
\]

can be displayed (not necessarily uniquely) as the product of matrices of simple shears.

*12. Diagonalize the quadratic form

\[ F = (a \cos^2 \theta + b \sin^2 \theta)x^2 + 2(a - b)(\sin \theta \cos \theta)xy + (a \sin^2 \theta + b \cos^2 \theta)y^2, \]

and identify the principal axes.

13. Find all eigenvalues, and an orthonormal set of eigenvectors, of the matrices

\[
A = \begin{pmatrix}
5 & 0 & \sqrt{3} \\
0 & 3 & 0 \\
\sqrt{3} & 0 & 3
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}.
\]

Hence sketch the surfaces

\[ 5x^2 + 3y^2 + 3z^2 + 2\sqrt{3}xz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 - xy - yz - zx = 1. \]

14. Let $\Sigma$ be the surface in $\mathbb{R}^3$ given by

\[ 2x^2 + 2xy + 4yz + z^2 = 1. \]

By writing this equation as

\[ x^T A x = 1, \]

with $A$ a real symmetric matrix, show that there is an orthonormal basis such that, if we use coordinates $(u, v, w)$ with respect to this new basis, $\Sigma$ takes the form

\[ \lambda u^2 + \mu v^2 + \nu w^2 = 1. \]

Find $\lambda$, $\mu$ and $\nu$ and hence find the minimum distance between the origin and $\Sigma$. \textbf{Hint: it is not necessary to find the basis explicitly.}

15. (i) Explain what is meant by saying that a $2 \times 2$ real matrix,

\[
A = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix},
\]

preserves the scalar product on $\mathbb{R}^2$ with respect to

(a) the Euclidean metric, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or (b) the Minkowski metric, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii) Using a single real parameter together with a choice of sign ($\pm 1$), give and justify a description of all matrices, $A$, that preserve the scalar product with respect to the Euclidean metric. Show that these matrices form a group.

(iii) Using a single real parameter together with a choice of sign ($\pm 1$), give and justify a description of all matrices, $A$ with $a > 0$, that preserve the scalar product with respect to the Minkowski metric. Show that these matrices form a group.

(iv) What is the intersection of the above two groups?
Revision Questions

16. Show that a rotation about the z-axis through an angle $\theta$ corresponds to the matrix

\[
R = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Write down a real eigenvector of $R$ and give the corresponding eigenvalue. In the case of a matrix corresponding to a general rotation, how can one find the axis of rotation?

A rotation through 45° about the x-axis is followed by a similar one about the z-axis. Show that the rotation corresponding to their combined effect has its axis inclined at equal angles to the $x$ and $z$ axes.

17. Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if $\lambda$ is an eigenvalue so is $\lambda^*$. Hence argue that the eigenvalues of a $3 \times 3$ real orthogonal matrix $Q$ must be a selection from $\pm 1$, $-1$ and $e^{i\alpha}$ & $e^{-i\alpha}$.

Verify that $\det Q = \pm 1$. What is the effect of $Q$ on vectors orthogonal to an eigenvector with eigenvalue $\pm 1$?

*18. This is another way of proving $\det AB = \det A \det B$. You may wish to stick to the case $n = 3$.

If $1 \leq r, s \leq n$, $r \neq s$ and $\lambda$ is real, let $E(\lambda, r, s)$ be an $n \times n$ matrix with $(i, j)$ entry $\delta_{ij} + \lambda \delta_{ir} \delta_{js}$. If $1 \leq r \leq n$ and $\mu$ is real, let $F(\mu, r)$ be an $n \times n$ matrix with $(i, j)$ entry $\delta_{ij} + (\mu - 1) \delta_{ir} \delta_{jr}$.

(i) Give a simple geometric interpretation of the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ associated with $E(\lambda, r, s)$ and $F(\mu, r)$.

(ii) Give a simple account of the effect of pre-multiplying an $n \times m$ matrix by $E(\lambda, r, s)$ and by $F(\mu, r)$. What is the effect of post-multiplying an $m \times n$ matrix?

(iii) If $A$ is an $n \times n$ matrix, find $\det(E(\lambda, r, s)A)$ and $\det(F(\mu, r)A)$ in terms of $\det A$.

(iv) Show that every $n \times n$ matrix is the product of matrices of the form $E(\lambda, r, s)$ and $F(\mu, r)$ and a diagonal matrix with entries 0 or 1.

(v) Use (iii) and (iv) to show that, if $A$ and $B$ are $n \times n$ matrices, then $\det A \det B = \det AB$.

*19. The object of this exercise is to show why finding eigenvalues of a large matrix is not just a matter of finding a large fast computer.

Consider the $n \times n$ complex matrix $A = \{A_{ij}\}$ given by

\[
A_{ij} = 1 \quad \text{for } 1 \leq j \leq n - 1, \\
A_{n1} = \kappa^n, \\
A_{ij} = 0 \quad \text{otherwise},
\]

where $\kappa \in \mathbb{C}$ is non-zero. Thus, when $n = 2$ and $n = 3$, we get the matrices

\[
\begin{pmatrix}
0 & 1 \\
\kappa^2 & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\kappa^2 & 0 & 0
\end{pmatrix}.
\]

(i) Find the eigenvalues and associated eigenvectors of $A$ for $n = 2$ and $n = 3$.

(ii) By guessing and then verifying your answers, or otherwise, find the eigenvalues and associated eigenvectors of $A$ for for all $n \geq 2$.

(iii) Suppose that your computer works to 15 decimal places and that $n = 100$. You decide to find the eigenvalues of $A$ in the cases $\kappa = 2^{-1}$ and $\kappa = 3^{-1}$. Explain why at least one (and more probably) both attempts will deliver answers which bear no relation to the true answers.