Exercises on Complex Numbers and an Exercise on Summation

These examples are designed to dust off a few cobwebs and check familiarity with material that the lecturer will assume. The examples are not intended as part of the Examples Sheets for supervision.

1. For \( z = a + ib \) and \( z^{-1} = \frac{a - ib}{a^2 + b^2} \) confirm that
   \[ zz^{-1} = 1 + i 0. \]

2. Confirm that
   (a) \( \bar{z} = z \);
   (b) \( z_1 + z_2 = \bar{z_1} + \bar{z_2} \);
   (c) \( \bar{z_1} \bar{z_2} = \bar{z_1} \bar{z_2} \);
   (d) \( \bar{(z^{-1})} = (\bar{z})^{-1} \).

3. Confirm that
   (a) \( |z|^2 = z \bar{z} \);
   (b) \( z^{-1} = \frac{\bar{z}}{|z|^2} \).

4. For \( x, y \in \mathbb{C} \) give a geometric interpretation of \( x/y \).

5. For \( x, y \in \mathbb{C} \) and \( n \) a positive integer, show that
   \[ \log(xy) = \log(x) + \log(y), \]
   and
   \[ \log(x^n) = n \log(x). \]
   You may assume any property of the complex exponential function required, and that \( \exp(\log(x)) = x \).

6. For non-zero \( w, z \in \mathbb{C} \) show that if \( zw - \bar{z}w = 0 \), then \( z = \gamma w \) for some \( \gamma \in \mathbb{R} \).

7. For positive integers \( m \) and \( n \) and coefficients \( a_{ij} \in \mathbb{C} \), with \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \), show that
   \[ \sum_{p=0}^{m} \sum_{q=0}^{n} a_{pq} = \sum_{q=0}^{n} \sum_{p=0}^{m} a_{pq} = \sum_{r=0}^{\min(r,m)} a_{p-r} = \sum_{r=0}^{\min(r,n)} \sum_{q=\max(0,r-m)}^{m+n} a_{r-q} q. \]  
   (1)

By taking
\[ a_{pq} = \frac{x^p y^q}{p! q!}, \]
for \( x, y \in \mathbb{C} \),
show that \( \exp(x) \exp(y) = \exp(x + y) \) on the assumption that, for this choice of \( a_{pq} \), (1) remains valid as \( m \to \infty \) and \( n \to \infty \).