

Exercises on Complex Numbers and an Exercise on Summation

These examples are designed to dust off a few cobwebs and check familiarity with material that the lecturer will assume. The examples are not intended as part of the Examples Sheets for supervision.

1. For $z = a + ib$ and $z^{-1} = \frac{a-ib}{a^2+b^2}$ confirm that

$$z z^{-1} = 1 + i \cdot 0.$$

2. Confirm that

(a)

$$\overline{\overline{z}} = z;$$

(b)

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2};$$

(c)

$$\overline{z_1 z_2} = \overline{z_1} \overline{z_2};$$

(d)

$$\overline{(z^{-1})} = (\overline{z})^{-1}.$$

3. Confirm that

(a)

$$|z|^2 = z \overline{z};$$

(b)

$$z^{-1} = \frac{\overline{z}}{|z|^2}.$$

4. For $x, y \in \mathbb{C}$ give a geometric interpretation of x/y .

5. For $x, y \in \mathbb{C}$ and n a positive integer, show that

$$\log(xy) = \log(x) + \log(y),$$

and

$$\log(x^n) = n \log(x).$$

You may assume any property of the complex exponential function required, and that $\exp(\log(x)) = x$.

6. For non-zero $w, z \in \mathbb{C}$ show that if $z\overline{w} - \overline{z}w = 0$, then $z = \gamma w$ for some $\gamma \in \mathbb{R}$.

7. For positive integers m and n and coefficients $a_{ij} \in \mathbb{C}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$, show that

$$\sum_{p=0}^m \sum_{q=0}^n a_{pq} = \sum_{q=0}^n \sum_{p=0}^m a_{pq} = \sum_{r=0}^{m+n} \sum_{p=\max(0, r-n)}^{\min(r, m)} a_{p \ r-p} = \sum_{r=0}^{m+n} \sum_{q=\max(0, r-m)}^{\min(r, n)} a_{r-q \ q}. \quad (1)$$

By taking

$$a_{pq} = \frac{x^p y^q}{p! q!}, \quad \text{for } x, y \in \mathbb{C},$$

show that $\exp(x)\exp(y) = \exp(x+y)$ on the *assumption* that, for this choice of a_{pq} , (1) remains valid as $m \rightarrow \infty$ and $n \rightarrow \infty$.