A1f **Vectors and Matrices: Study Sheet** Michaelmas 2022

Exercises on Suffix Notation and the Summation Convention

These exercises are designed to help to familiarise students with this topic and are not intended as part of the Examples Sheets for supervision.

Free suffices and dummy suffices are assumed to range/sum through $1, 2, \ldots, n$ unless stated otherwise.

1. Write each of the following using the summation convention:

   (a) $d\theta = \frac{\partial\theta}{\partial x_1}\, dx_1 + \frac{\partial\theta}{\partial x_2}\, dx_2 + \ldots + \frac{\partial\theta}{\partial x_n}\, dx_n$;

   (b) $\frac{d\overline{x}_k}{dt} = \frac{\partial\overline{x}_k}{\partial x_1}\,\frac{dx_1}{dt} + \frac{\partial\overline{x}_k}{\partial x_2}\,\frac{dx_2}{dt} + \ldots + \frac{\partial\overline{x}_k}{\partial x_n}\,\frac{dx_n}{dt}$;

   (c) $(x_1)^2 + (x_2)^2 + (x_3)^2 + \ldots + (x_n)^2$;

   (d) $\displaystyle\sum_{p=1}^{3}\sum_{q=1}^{3} g_{pq} dx_p dx_q$.

2. Write the following expressions in full:

   (a) $a_{jk}x_k$;

   (b) $A_{pq}A_{qr}$;

   (c) $\frac{\partial}{\partial x_k}(\sqrt{g}A_k)$ with $n = 3$;

   (d) $A_{jk}B_{pk}C_j$ with $n = 2$.

3. Write each of the following using the summation convention.

   (a) $a_1x_1x_3 + a_2x_2x_3 + \ldots + a_nx_nx_3$;

   (b) $A_{21}B_1 + A_{22}B_2 + A_{23}B_3 + \ldots + A_{2n}B_n$;

   (c) $g_{21}g_{11} + g_{22}g_{21} + g_{23}g_{31} + g_{24}g_{41}$.

4. If $n = 2$, write out the system of equations represented by $a_{pq}x_q = b_p$.

From now on assume $n = 3$ unless stated otherwise.

5. If

$$\begin{array}{llllll} a_{11} = 1\,, & a_{12} = -1\,, & a_{13} = 0\,, \\ a_{21} = -2\,, & a_{22} = 3\,, & a_{23} = 1\,, \\ a_{31} = 2\,, & a_{32} = 0\,, & a_{33} = 4\,, \end{array}$$

   show that

$$\begin{array}{lll} a_{ii} = 8\,, & a_{i1}a_{i2} = -7\,, & a_{i2}a_{i3} = 3\,, \\ a_{1i}a_{2i} = -5\,, & a_{2i}a_{3i} = 0\,, & a_{i1}a_{2i} = -6\,. \end{array}$$

6. If the numbers a_{ij} are as given in question (5) above and if $b_1 = 1$, $b_2 = -1$ and $b_3 = 4$, show that

$$a_{1i}b_i = 2\,, \quad a_{j1}b_j = 11\,, \quad a_{ji}a_{i1}b_j = 49\,.$$

   Hint: for the last part, first evaluate $a_{j1}b_j, a_{j2}b_j$ and $a_{j3}b_j$.

7. Show that $\delta_{ij}b_j = \delta_{ji}b_j = b_i$.

8. If the numbers a_{ij} are as given in (5) above, evaluate

$$\text{(a) } a_{1j}\delta_{1j}\,, \quad \text{(b) } a_{12}\delta_{ii}\,, \quad \text{(c) } a_{1i}a_{2k}\delta_{ik}\,.$$

9. Assume the suffix i takes all integral values from 0 to ∞, and that a_i and b_i are defined by

$$a_i = x^i\,, \quad \text{and} \quad b_i = \frac{1}{i!}$$

   respectively, where x is a constant and, by definition, $0! = 1$. Show that

$$a_i\, b_i = e^x\,.$$

10. If the quantities e_{ij}, and e'_{ij} satisfy the relation

$$l_{\ell i}\, l_{mj}\, e_{ij} = e'_{\ell m}\,,$$

and if

$$l_{ki}\, l_{kj} = \delta_{ij}\,,$$

show that

$$e_{ij} = l_{\ell i} l_{mj} e'_{\ell m}\,.$$

Hint: multiply the first equation by $l_{\ell p} l_{mq}$.

11. Without referring to your notes, prove that

$$\delta_{ij}\epsilon_{ijk} = 0\,, \quad \epsilon_{ijk}\epsilon_{rjk} = 2\delta_{ir} \quad \text{and} \quad \epsilon_{ijk}\epsilon_{ijk} = 6\,.$$

Answers

1. (a) $d\theta = \frac{\partial\theta}{\partial x_j}\, dx_j$;
   (b) $\frac{d\overline{x}_k}{dt} = \frac{\partial\overline{x}_k}{\partial x_m}\,\frac{dx_m}{dt}$;
   (c) $x_k x_k$;
   (d) $g_{pq} dx_p dx_q$ with $n = 3$.

2. (a)
$$\sum_{k=1}^{n} a_{jk}x_k = a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n\,;$$
   (b)
$$\sum_{q=1}^{n} A_{pq}A_{qr} = A_{p1}A_{1r} + A_{p2}A_{2r} + \ldots + A_{pn}A_{nr}\,;$$
   (c)
$$\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(\sqrt{g}A_k) = \frac{\partial}{\partial x_1}(\sqrt{g}A_1) + \frac{\partial}{\partial x_2}(\sqrt{g}A_2) + \frac{\partial}{\partial x_3}(\sqrt{g}A_3)\,;$$
   (d)
$$\begin{aligned}\sum_{j,k=1}^{2} A_{jk}B_{pk}C_j &= \sum_{k=1}^{2}(A_{1k}B_{pk}C_1 + A_{2k}B_{pk}C_2)\\ &= A_{11}B_{p1}C_1 + A_{12}B_{p2}C_1 + A_{21}B_{p1}C_2 + A_{22}B_{p2}C_2\,.\end{aligned}$$

3. (a) $a_k x_k x_3$;
   (b) $A_{2j}B_j$;
   (c) $g_{2p}g_{p1}$ with $n = 4$.

8. (a) 1;
   (b) -3;
   (c) -5.