## Examples Sheet 1

*The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets.*

**1**. Show, from first principles, that, for non-negative integer $n$, $dx^n/dx = nx^{n-1}$.

**2**. Let $f(x) = u(x)v(x)$. Use the definition of the derivative of a function to show that

$$\frac{df}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v\,.$$

**3**. Calculate:

(i) $d^n(xe^x)/dx^n$ using (a) the Leibniz rule and (b) application of the product rule;

(ii) $d^3(\ln x)^2/dx^3$.

**4**. (i) Write down or determine the Taylor series for $f(x) = e^{ax}$ about $x = 1$.

(ii) Write down or determine the Taylor series for $\ln(1+x)$ about $x = 0$. Then show that

$$\lim_{k\to\infty} k\ln\left(1+\frac{x}{k}\right) = x$$

and deduce that

$$\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k = e^x\,.$$

**5**. Determine by any method the first three non-zero terms of the Taylor expansions about $x = 0$ of:

(i) $(x^2+a)^{-3/2}$;

(ii) $\ln(\cos x)$;

(iii) $\exp\left[-(x-a)^{-2}\right]$, where $a$ is a constant.

**6**. By considering the area under the curves $y = \ln x$ and $y = \ln(x-1)$, show that

$$N \ln N - N < \ln(N!) < (N+1)\ln(N+1) - N.$$

Hence show that

$$\frac{N!}{N+1} < \left(\frac{N+1}{e}\right)^N .$$

**7**. Show that

$$y(x) = \int_x^\infty e^{-t^2}\,dt$$

satisfies the differential equation $y'' + 2xy' = 0$.

$*$**8**. Let $J_n$ be the indefinite integral

$$J_n = \int \frac{x^{-n}}{(ax^2+2bx+c)^{\frac{1}{2}}}\,dx\,.$$

By integrating $\int x^{-(n+1)}(ax^2+2bx+c)^{\frac{1}{2}}\,dx$ by parts, show that, for $n \neq 0$,

$$ncJ_{n+1} + (2n-1)bJ_n + (n-1)aJ_{n-1} = -x^{-n}(ax^2+2bx+c)^{\frac{1}{2}}\,.$$

Hence evaluate

$$\int_1^2 \frac{1}{x^{5/2}(x+2)^{\frac{1}{2}}}\,dx\,.$$

$*$**9**. In a large population, the proportion with income between $x$ and $x+dx$ is $f(x)dx$. Express the mean (average) income $\mu$ as an integral, assuming that any positive income is possible.

Let $p = F(x)$ be the proportion of the population with income less than $x$, and $G(x)$ be the mean (average) income earned by people with income less than $x$. Further, let $\theta(p)$ be the proportion of the total income that is earned by people with income less than $x$ as a function of the proportion $p$ of the population that has income less than $x$. Express $F(x)$ and $G(x)$ as integrals and thence derive an expression for $\theta(p)$, showing that

$$\theta(0) = 0\,, \qquad \theta(1) = 1$$

and

$$\theta'(p) = \frac{F^{-1}(p)}{\mu}\,, \qquad \theta''(p) = \frac{1}{\mu f(F^{-1}(p))} > 0\,.$$

Sketch the graph of a function $\theta(p)$ with these properties. Deduce that, if there is any variation in income, the bottom (when ordered in terms of income) proportion $p$ of the population receive less than $p$ of the total income, for all positive values of $p$. Just how much less is quantified by the (in)famous "Gini index" beloved of economists, which is twice the area between the curve $\theta(p)$ and the diagonal line connecting $(0,0)$ and $(1,1)$.

A particular population's income is described by the exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0\,.$$

For some constant $\lambda > 0$, using your expression for $\theta(p)$, compute the Gini index in this case.

[*Food for thought: To what extent does the Gini index capture the degree of income inequality for this distribution?*]

**10**. For $f(x, y) = \exp(-xy)$, find $(\partial f/\partial x)|_y$ and $(\partial f/\partial y)|_x$. Check that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}\,.$$

Now express $x$ and $y$ in polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. Find $(\partial f/\partial r)|_\theta$ and $(\partial f/\partial \theta)|_r$ using:

(a) the chain rule; and

(b) by first expressing $f$ in terms of $r$ and $\theta$.

Check that the two methods give the same results.

**11**. If $xyz + x^3 + y^4 + z^5 = 0$ (an implicit equation for any of the variables $x, y, z$ in terms of the other two), find

$$\left.\frac{\partial x}{\partial y}\right|_z, \quad \left.\frac{\partial y}{\partial z}\right|_x \quad \text{and} \quad \left.\frac{\partial z}{\partial x}\right|_y$$

and show that their product is $-1$.

Does this result hold for an arbitrary relation $f(x, y, z) = 0$?

What about $f(x_1, x_2, \ldots, x_n) = 0$?

**12**. In thermodynamics, the pressure of a system, $p$, can be considered as a function of the variables $V$ (volume) and $T$ (temperature) or as a function of $V$ and $S$ (entropy).

(i) By expressing $p(V, S)$ in the form $p(V, S(V, T))$ evaluate

$$\left.\frac{\partial p}{\partial V}\right|_T - \left.\frac{\partial p}{\partial V}\right|_S \quad \text{in terms of} \quad \left.\frac{\partial S}{\partial V}\right|_T \quad \text{and} \quad \left.\frac{\partial S}{\partial p}\right|_V\,.$$

(ii) Hence, using $TdS = dU + pdV$ (conservation of energy with $U$ the internal energy), show that

$$\left.\frac{\partial \ln p}{\partial \ln V}\right|_T - \left.\frac{\partial \ln p}{\partial \ln V}\right|_S = \left.\frac{\partial (pV)}{\partial T}\right|_V \left[\frac{p^{-1}\,(\partial U/\partial V)|_T + 1}{(\partial U/\partial T)|_V}\right].$$

[*Hint: note that* $(\partial \ln p/\partial \ln V)|_T = (V/p)\,(\partial p/\partial V)|_T$.]

**13**. By differentiating $I$ with respect to $\lambda$, show that

$$I(\lambda, \alpha) = \int_0^\infty \frac{\sin \lambda x}{x} e^{-\alpha x}\, dx = \tan^{-1}\left(\frac{\lambda}{\alpha}\right) + c(\alpha)\,.$$

Show that $c(\alpha)$ is constant (independent of $\alpha$) and hence, by considering the limits $\alpha \to \infty$ and $\alpha \to 0$, show that, if $\lambda > 0$,

$$\int_0^\infty \frac{\sin \lambda x}{x}\, dx = \frac{\pi}{2}\,.$$

What is the value of the integral when $\lambda < 0$?

**14**. Let

$$f(x) = \left(\int_0^x e^{-t^2}\, dt\right)^2 \quad \text{and} \quad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{(1+t^2)}\, dt\,.$$

Show that

$$f'(x) + g'(x) = 0\,.$$

Deduce that

$$f(x) + g(x) = \pi/4\,,$$

and hence that

$$\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}\,.$$

*Comments and corrections may be sent by email to a.d.challinor@ast.cam.ac.uk*