1. According to Newton’s law of cooling, the rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A forensic scientist enters a crime scene at 5:00 pm and discovers a cup of tea at temperature 40°C. At 5:30 pm its temperature is only 30°C. Giving all details of the mathematical methodology employed and assumptions made, estimate the time at which the tea was made.

2. Determine the half-life of Thorium-234 if a sample of 5 grams is reduced to 4 grams in one week. What amount of Thorium is left after three months?

3. Find the solutions of the initial value problems
   (i) \( y' + 2y = e^{-x}, \quad y(0) = 1 \);
   (ii) \( y' - y = 2xe^{2x}, \quad y(0) = 1 \).

4. Show that the general solution of
   \[ y' - y = e^{ux}, \quad u \neq 1, \]  
   \((*)\)
   can be written (by means of a suitable choice of \(A\)) in the form
   \[ y(x) = Ae^x + \frac{e^{ux} - e^x}{u - 1}. \]
   By taking the limit as \(u \to 1\) and using l’Hôpital’s rule, find the general solution of (*) when \(u = 1\).

5. Solve
   (i) \( y'x \sin x + (\sin x + x \cos x)y = xe^x \);
   (ii) \( y' \tan x + y = 1 \);
   (iii) \( y' = (e^y - x)^{-1} \).
6. Find the general solutions of
   (i) \( y' = x^2(1 + y^2) \),
   (ii) \( y' = \cos^2 x \cos^2 2y \),
   (iii) \( y' = (x - y)^2 \),
   (iv) \( (e^y + x)y' + (e^x + y) = 0 \).

7. Find all solutions of the equation
   \[ y \frac{dy}{dx} - x = 0, \]
   and give a sketch showing the solutions. By means of the substitution \( y = \log u - x \), deduce the general solution of
   \[ (\log u - x) \frac{du}{dx} - u \log u = 0. \]
   Sketch the solutions, starting from your previous sketch and drawing first the lines to which \( y = \pm x \) are mapped.

8. In each of the following sketch a few solution curves. It might help you to consider values of \( y' \) on the axes, or contours of constant \( y' \), or the asymptotic behaviour when \( y \) is large.
   (i) \( y' + xy = 1 \),
   (ii) \( y' = x^2 + y^2 \),
   (iii) \( y' = (1 - y)(2 - y) \).

9. (i) Sketch the solution curves for the equation
    \[ \frac{dy}{dx} = xy. \]
    Find the family of solutions determined by this equation and reassure yourself that your sketches were appropriate.

(ii) Sketch the solution curves for the equation
    \[ \frac{dy}{dx} = \frac{x - y}{x + y}. \]
    By rewriting the equation in the form
    \[ \left( x \frac{dy}{dx} + y \right) + y \frac{dy}{dx} = x, \]
    find and sketch the family of solutions.

*Does the substitution \( y = ux \) lead to an easier method of solving this equation?
10. Measurements on a yeast culture have shown that the rate of increase of the amount, or ‘biomass’, of yeast is related to the biomass itself by the equation

\[ \frac{dN}{dt} = aN - bN^2 , \]

where \( N(t) \) is a measure of the biomass at time \( t \), and \( a \) and \( b \) are positive constants. Without solving the equation, find in terms of \( a \) and \( b \):

(i) the value of \( N \) at which \( \frac{dN}{dt} \) is a maximum;
(ii) the values of \( N \) at which \( \frac{dN}{dt} \) is zero.

Using this information, sketch the graph of \( N(t) \) against \( t \), and compare this with what you obtain by solving the equation analytically for \( 0 \leq N \leq a/b \).

11. Water flows into a cylindrical bucket of depth \( H \) and cross-sectional area \( A \) at a volume flow rate \( Q \) which is constant. There is a hole in the bottom of the bucket of cross-sectional area \( a \ll A \). When the water level above the hole is \( h \), the flow rate out of the hole is \( a\sqrt{2gh} \), where \( g \) is the gravitational acceleration. Derive an equation for \( \frac{dh}{dt} \). Find the equilibrium depth \( h_e \) of water, and show that it is stable.

12. In each of the following equations for \( y(t) \), find the equilibrium points and classify their stability properties:

(i) \[ \frac{dy}{dt} = y(y - 1)(y - 2) , \]
(ii) \[ \frac{dy}{dt} = -2 \tan^{-1}[y/(1 + y^2)] , \]
*(iii) \[ \frac{dy}{dt} = y^3(e^y - 1)^2 . \]

13. Investigate the stability of the constant solutions \( (u_{n+1} = u_n) \) of the discrete equation

\[ u_{n+1} = 4u_n(1 - u_n) . \]

In the case \( 0 \leq u_0 \leq 1 \), use the substitution \( u_0 = \sin^2 \theta \) to find the general solution and verify your stability results. Can you find an explicit form of the general solution in the case \( u_0 > 1 \)?

*14. Two identical snowploughs plough the same stretch of road in the same direction. The first starts at \( t = 0 \) when the depth of snow is \( h_0 \) and the second starts from the same point \( T \) seconds later. Snow falls so that the depth of snow increases at a constant rate of \( k \) ms\(^{-1} \). The speed of each snowplough is \( k/(ah) \) where \( h \) is the depth of snow it is ploughing and \( a \) is a constant, and each snowplough clears all the snow. Show that the time taken for the first snowplough to travel \( x \) metres is

\[ (e^{ax} - 1)h_0k^{-1} \text{ seconds}. \]
Show also that the time $t$ by which the second snowplough has travelled $x$ metres satisfies the equation

$$\frac{1}{a} \frac{dt}{dx} = t - (e^{ax} - 1)h_0 k^{-1}.$$  

Hence show that the snowploughs will collide when they have moved a distance $kT/(ah_0)$ metres.

Comments and corrections may be sent by email to J.R.Taylor@damtp.cam.ac.uk