1. According to Newton’s law of cooling, the rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A forensic scientist enters a crime scene at 5:00 pm and discovers a cup of tea at temperature $40^\circ C$. At 5:30 pm its temperature is only $30^\circ C$. Giving all details of the mathematical methodology employed and assumptions made, estimate the time at which the tea was made.

2. Determine the half-life of Thorium-234 if a sample of 5 grams is reduced to 4 grams in one week. What amount of Thorium is left after three months?

3. Find the solutions of the initial value problems
   
   (i) $y' + 2y = e^{-x}$, $y(0) = 1$;
   (ii) $y' - y = 2xe^{2x}$, $y(0) = 1$.

4. Show that the general solution of
   
   $$y' - y = e^{ux} , \quad u \neq 1$$

   can be written (by means of a suitable choice of $A$) in the form
   
   $$y(x) = Ae^x + \frac{e^{ux} - e^x}{u - 1}.$$ 

   By taking the limit as $u \to 1$ and using l’Hôpital’s rule, find the general solution of (*) when $u = 1$.

5. Solve
   
   (i) $y'x \sin x + (\sin x + x \cos x)y = xe^x$;
   (ii) $y' \tan x + y = 1$;
   (iii) $y' = (e^y - x)^{-1}$. 
6. Find the general solutions of
   (i) \( y' = x^2(1 + y^2) \),
   (ii) \( y' = \cos^2 x \cos^2 2y \),
   (iii) \( y' = (x - y)^2 \),
   (iv) \((e^y + x)y' + (e^x + y) = 0\).

7. Find all solutions of the equation
   \[ y \frac{dy}{dx} - x = 0, \]
   and give a sketch showing the solutions. By means of the substitution \( y = \log u - x \), deduce
   the general solution of
   \[ (\log u - x) \frac{du}{dx} - u \log u = 0. \]
   Sketch the solutions, starting from your previous sketch and drawing first the lines to which
   \( y = \pm x \) are mapped.

8. In each of the following sketch a few solution curves. It might help you to consider values of \( y' \)
on the axes, or contours of constant \( y' \), or the asymptotic behaviour when \( y \) is large.
   (i) \( y' + xy = 1 \),
   (ii) \( y' = x^2 + y^2 \),
   (iii) \( y' = (1 - y)(2 - y) \).

9. (i) Sketch the solution curves for the equation
   \[ \frac{dy}{dx} = xy. \]
   Find the family of solutions determined by this equation and reassure yourself that your
   sketches were appropriate.
   (ii) Sketch the solution curves for the equation
   \[ \frac{dy}{dx} = \frac{x - y}{x + y}. \]
   By rewriting the equation in the form
   \[ \left( x \frac{dy}{dx} + y \right) + y \frac{dy}{dx} = x, \]
   find and sketch the family of solutions.
   *Does the substitution \( y = ux \) lead to an easier method of solving this equation?
10. Measurements on a yeast culture have shown that the rate of increase of the amount, or ‘biomass’, of yeast is related to the biomass itself by the equation
\[
\frac{dN}{dt} =aN - bN^2 ,
\]
where \(N(t)\) is a measure of the biomass at time \(t\), and \(a\) and \(b\) are positive constants. Without solving the equation, find in terms of \(a\) and \(b\):
(i) the value of \(N\) at which \(dN/dt\) is a maximum;
(ii) the values of \(N\) at which \(dN/dt\) is zero, and the corresponding values of \(d^2N/dt^2\).
Using all this information, sketch the graph of \(N(t)\) against \(t\), and compare this with what you obtain by solving the equation analytically for \(0 \leq N \leq a/b\).

11. Water flows into a cylindrical bucket of depth \(H\) and cross-sectional area \(A\) at a volume flow rate \(Q\) which is constant. There is a hole in the bottom of the bucket of cross-sectional area \(a \ll A\). When the water level above the hole is \(h\), the flow rate out of the hole is \(a\sqrt{2gh}\), where \(g\) is the gravitational acceleration. Derive an equation for \(dh/dt\). Find the equilibrium depth \(h_e\) of water, and show that it is stable.

12. In each of the following equations for \(y(t)\), find the equilibrium points and classify their stability properties:
(i) \(\frac{dy}{dt} = y(y - 1)(y - 2)\),
(ii) \(\frac{dy}{dt} = -2 \tan^{-1}[y/(1 + y^2)]\),
*(iii) \(\frac{dy}{dt} = y^3(e^y - 1)^2\).

13. Investigate the stability of the constant solutions \((u_{n+1} = u_n)\) of the discrete equation
\[
 u_{n+1} = 4u_n(1 - u_n).
\]
In the case \(0 \leq u_0 \leq 1\), use the substitution \(u_0 = \sin^2 \theta\) to find the general solution and verify your stability results. Can you find an explicit form of the general solution in the case \(u_0 > 1\)?

*14. Two identical snowploughs plough the same stretch of road in the same direction. The first starts at \(t = 0\) when the depth of snow is \(h_0\) and the second starts from the same point \(T\) seconds later. Snow falls so that the depth of snow increases at a constant rate of \(k\) ms\(^{-1}\). The speed of each snowplough is \(k/(ah)\) where \(h\) is the depth of snow it is ploughing and \(a\) is a constant, and each snowplough clears all the snow. Show that the time taken for the first snowplough to travel \(x\) metres is
\[
(e^{ax} - 1)h_0k^{-1} \text{ seconds}.
\]
Show also that the time $t$ by which the second snowplough has travelled $x$ metres satisfies the equation

$$\frac{1}{a} \frac{dt}{dx} = t - (e^{ax} - 1)h_0 k^{-1}.$$

Hence show that the snowploughs will collide when they have moved a distance $kT/(ah_0)$ metres.

Comments and corrections may be sent by email to J.R.Taylor@damtp.cam.ac.uk