1. According to Newton’s law of cooling, the rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A forensic scientist enters a crime scene at 5:00 pm and discovers a cup of tea at temperature $40^\circ$C. At 5:30 pm its temperature is only $30^\circ$C. Giving all details of the mathematical methodology employed and assumptions made, estimate the time at which the tea was made.

2. Determine the half-life of thorium-234 if a sample of mass 5 g is reduced to 4 g in one week. What amount of thorium is left after three months?

3. Find the solutions of the initial-value problems:
   (i) $y' + 2y = e^{-x}$, $y(0) = 1$;
   (ii) $y' - y = 2xe^{2x}$, $y(0) = 1$.

4. Show that the general solution of
   \[ y' - y = e^{ux}, \tag{*} \]
   for $u \neq 1$, can be written (by means of a suitable choice of $A$) in the form
   \[ y(x) = A e^x + \frac{e^{ux} - e^x}{u - 1}. \]
   By taking the limit as $u \to 1$ and using L’Hôpital’s rule, find the general solution of (*) when $u = 1$.

5. Solve:
   (i) $x \sin x \, y' + (\sin x + x \cos x) y = xe^x$;
   (ii) $\tan x \, y' + y = 1$;
   (iii) $y' = (e^y - x)^{-1}$.
6. Find the general solutions of:
   (i) \( y' = x^2(1 + y^2) \);
   (ii) \( y' = \cos^2 x \cos^2 2y \);
   (iii) \( y' = (x - y)^2 \);
   (iv) \((e^y + x)y' + (e^x + y) = 0\).

7. Find all solutions of the equation

\[
y \frac{dy}{dx} - x = 0,
\]

and give a sketch showing the solutions. By means of the substitution \( y = \ln u - x \),

deduce the general solution of

\[
(\ln u - x) \frac{du}{dx} - u \ln u = 0.
\]

Sketch the solutions, starting from your previous sketch and drawing first the lines to

which \( y = \pm x \) are mapped.

8. In each of the following cases, sketch a few solution curves:
   (i) \( y' + xy = 1 \);
   (ii) \( y' = x^2 + y^2 \);
   (iii) \( y' = (1 - y)(2 - y) \).

It might help you to consider values of \( y' \) on the axes, or contours of constant \( y' \), or the

asymptotic behaviour when \( y \) is large.

9. (i) Sketch the solution curves for the equation

\[
\frac{dy}{dx} = xy.
\]

Find the family of solutions determined by this equation and reassure yourself that

your sketches were appropriate.

(ii) Sketch the solution curves for the equation

\[
\frac{dy}{dx} = \frac{x - y}{x + y}.
\]

By rewriting the equation in the form

\[
\left( x \frac{dy}{dx} + y \right) + y \frac{dy}{dx} = x,
\]

find and sketch the family of solutions.

*Does the substitution \( y = ux \) lead to an easier method of solving (*)?
10. Measurements on a yeast culture have shown that the rate of increase of the amount, or ‘biomass’, of yeast is related to the biomass itself by the equation

\[ \frac{dN}{dt} = aN - bN^2, \]

where \( N(t) \) is a measure of the biomass at time \( t \), and \( a \) and \( b \) are positive constants. Without solving the equation, find in terms of \( a \) and \( b \):

(i) the value of \( N \) at which \( dN/dt \) is a maximum; and

(ii) the values of \( N \) at which \( dN/dt \) is zero.

Using this information, sketch graphs of \( N(t) \) against \( t \), and compare these with what you obtain by solving the equation analytically for \( 0 \leq N \leq a/b \).

11. Water flows into a deep cylindrical bucket of cross-sectional area \( A \) at a volume flow rate \( Q \), which is constant. There is a hole in the bottom of the bucket of cross-sectional area \( a \ll A \). When the water level above the hole is \( h \), the volume flow rate out of the hole is \( a\sqrt{2gh} \), where \( g \) is the gravitational acceleration. Derive an equation for \( dh/dt \). Find the equilibrium depth \( h_e \) of water, and show that it is stable.

12. In each of the following equations for \( y(t) \), find the equilibrium points and classify their stability properties:

(i) \( \frac{dy}{dt} = y(y - 1)(y - 2) \);

(ii) \( \frac{dy}{dt} = -2 \tan^{-1}\left[\frac{y}{1 + y^2}\right] \);

* (iii) \( \frac{dy}{dt} = y^3(e^y - 1)^2 \).

13. Investigate the stability of the constant solutions \((u_{n+1} = u_n)\) of the discrete equation

\[ u_{n+1} = 4u_n(1 - u_n). \]

In the case \( 0 \leq u_0 \leq 1 \), use the substitution \( u_0 = \sin^2 \theta \) to find the general solution and verify your stability results. Can you find an explicit form of the general solution in the case \( u_0 > 1 \)?

*14. Two identical snowploughs plough the same stretch of road in the same direction. The first starts at \( t = 0 \) when the (uniform) depth of snow is \( h_0 \) and the second starts from the same point at time \( T \) later. Snow falls so that the depth of snow increases at a constant rate \( k \). The speed of each snowplough is \( k/(ah) \), where \( h \) is the depth of snow it is ploughing and \( a \) is a constant, and each snowplough clears all the snow. Show that the time taken for the first snowplough to travel a distance \( x \) from its starting point, \( t_1(x) \), is

\[ t_1(x) = \frac{h_0}{k} \left( e^{ax} - 1 \right). \]
Show also that the time $t_2(x)$ by which the second snowplough has travelled a distance $x$ satisfies the equation

$$\frac{1}{a} \frac{dt_2}{dx} = t_2 - \frac{h_0}{k} (e^{ax} - 1).$$

Hence show that the snowploughs will collide when they have moved a distance $kT/(ah_0)$.

Comments and corrections may be sent by email to a.d.challinor@ast.cam.ac.uk