Mathematical Tripos Part IA

C. P. Caulfield

Differential Equations A3

Michaelmas 2016

Examples Sheet 2

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on earlier sheets

1. According to Newton's law of cooling, the rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A forensic scientist enters a crime scene at 5:00 pm and discovers a cup of tea at temperature 40°C. At 5:30 pm its temperature is only 30°C. Giving all details of the mathematical methodology employed and assumptions made, estimate the time at which the tea was made.

2. Determine the half-life of Thorium-234 if a sample of 5 grams is reduced to 4 grams in one week. What amount of Thorium is left after three months?

3. Find the solutions of the initial value problems

   (i) $y' + 2y = e^{-x}$, $y(0) = 1$;

   (ii) $y' - y = 2xe^{2x}$, $y(0) = 1$.

4. Show that the general solution of

   $$y' - y = e^{ux} \ , \ u \neq 1 \ , \qquad (*)$$

   can be written (by means of a suitable choice of A) in the form

   $$y(x) = Ae^x + \frac{e^{ux} - e^x}{u - 1} \ .$$

   By taking the limit as $u \to 1$ and using l'Hôpital's rule, find the general solution of (*) when $u = 1$.

5. Solve

   (i) $y'x \sin x + (\sin x + x \cos x)y = xe^x$;

   (ii) $y' \tan x + y = 1$;

   (iii) $y' = (e^y - x)^{-1}$.

6. Find the general solutions of

(i) $y' = x^2(1+y^2)$,

(ii) $y' = \cos^2 x \cos^2 2y$,

(iii) $y' = (x-y)^2$,

(iv) $(\mathrm{e}^y + x)y' + (\mathrm{e}^x + y) = 0$.

7. Find all solutions of the equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} - x = 0 ,$$

and give a sketch showing the solutions. By means of the substitution $y = \log u - x$, deduce the general solution of

$$(\log u - x)\frac{\mathrm{d}u}{\mathrm{d}x} - u\log u = 0 .$$

Sketch the solutions, starting from your previous sketch and drawing first the lines to which $y = \pm x$ are mapped.

8. In each of the following sketch a few solution curves. It might help you to consider values of y' on the axes, or contours of constant y', or the asymptotic behaviour when y is large.

(i) $y' + xy = 1$,

(ii) $y' = x^2 + y^2$,

(iii) $y' = (1-y)(2-y)$.

9. (i) Sketch the solution curves for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy .$$

Find the family of solutions determined by this equation and reassure yourself that your sketches were appropriate.

(ii) Sketch the solution curves for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+y} .$$

By rewriting the equation in the form

$$\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + y\frac{\mathrm{d}y}{\mathrm{d}x} = x ,$$

find and sketch the family of solutions.

*Does the substitution $y = ux$ lead to an easier method of solving this equation?

10. Measurements on a yeast culture have shown that the rate of increase of the amount, or 'biomass', of yeast is related to the biomass itself by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN - bN^2 ,$$

where $N(t)$ is a measure of the biomass at time t, and a and b are positive constants. Without solving the equation, find in terms of a and b:

(i) the value of N at which $\mathrm{d}N/\mathrm{d}t$ is a maximum;

(ii) the values of N at which $\mathrm{d}N/\mathrm{d}t$ is zero, and the corresponding values of $\mathrm{d}^2N/\mathrm{d}t^2$.

Using all this information, sketch the graph of $N(t)$ against t, and compare this with what you obtain by solving the equation analytically for $0 \le N \le a/b$.

11. Water flows into a cylindrical bucket of depth H and cross-sectional area A at a volume flow rate Q which is constant. There is a hole in the bottom of the bucket of cross-sectional area $a \ll A$. When the water level above the hole is h, the flow rate out of the hole is $a\sqrt{2gh}$, where g is the gravitational acceleration. Derive an equation for $\mathrm{d}h/\mathrm{d}t$. Find the equilibrium depth h_e of water, and show that it is stable.

12. In each of the following equations for $y(t)$, find the equilibrium points and classify their stability properties:

(i) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)(y-2)$,

(ii) $\frac{\mathrm{d}y}{\mathrm{d}t} = -2\tan^{-1}[y/(1+y^2)]$,

*(iii) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3(\mathrm{e}^y - 1)^2$.

13. Investigate the stability of the constant solutions ($u_{n+1} = u_n$) of the discrete equation

$$u_{n+1} = 4u_n(1-u_n).$$

In the case $0 \le u_0 \le 1$, use the substitution $u_0 = \sin^2\theta$ to find the general solution and verify your stability results. Can you find an explicit form of the general solution in the case $u_0 > 1$?

*14. Two identical snowploughs plough the same stretch of road in the same direction. The first starts at $t = 0$ when the depth of snow is h_0 and the second starts from the same point T seconds later. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. The speed of each snowplough is $k/(ah)$ where h is the depth of snow it is ploughing and a is a constant, and each snowplough clears all the snow. Show that the time taken for the first snowplough to travel x metres is

$$(\mathrm{e}^{ax} - 1)h_0k^{-1} \text{ seconds.}$$

Show also that the time t by which the second snowplough has travelled x metres satisfies the equation

$$\frac{1}{a}\frac{\mathrm{d}t}{\mathrm{d}x} = t - (\mathrm{e}^{ax} - 1)h_0 k^{-1} \ .$$

Hence show that the snowploughs will collide when they have moved a distance $kT/(ah_0)$ metres.

Comments and corrections may be sent by email to cpc12@cam.ac.uk