**Mathematical Tripos Part IA** **Michaelmas Term 2022**

**Differential Equations** **Prof. Anthony Challinor**

# Examples Sheet 3

*The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets.*



**1**. Find the general solutions of:

(i) $y'' + 5y' + 6y = e^{3x}$ ;

(ii) $y'' + 9y = \cos 3x$ ;

(iii) $y'' - 2y' + y = (x-1)e^x$ .

**2**. The function $y(x)$ satisfies the linear equation

$$y'' + p(x)y' + q(x)y = 0\,.$$

The Wronskian $W(x)$ of two independent solutions, denoted $y_1(x)$ and $y_2(x)$, is defined to be

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\,.$$

Let $y_1(x)$ be given. Use the Wronskian to determine a first-order inhomogeneous differential equation for $y_2(x)$. Hence, show that

$$y_2(x) = y_1(x)\int_{x_0}^{x} \frac{W(t)}{[y_1(t)]^2}\,dt\,. \qquad (*)$$

Show that $W(x)$ satisfies

$$\frac{dW}{dx} + p(x)W = 0\,.$$

Verify that $y_1(x) = 1 - x$ is a solution of

$$xy'' - (1-x^2)y' - (1+x)y = 0\,. \qquad (\dagger)$$

Hence, using $(*)$ with $x_0 = 0$ and expanding the integrand in powers of $t$ to order $t^3$, find the first three non-zero terms in the power-series expansion for a solution, $y_2$, of $(\dagger)$ that is independent of $y_1$ and satisfies $y_2(0) = 0$, $y_2''(0) = 1$.

**3**. Find the general solutions of the following discrete equations:

(i) $y_{n+2} + y_{n+1} - 6y_n = n^2$ ;

(ii) $y_{n+2} - 3y_{n+1} + 2y_n = n$ ;

(iii) $y_{n+2} - 4y_{n+1} + 4y_n = a^n$ , where $a \neq 2$.

For (iii), by expressing $a^n$ as a Taylor series about $a = 2$, find the general solution in the case $a = 2$.

**4**. (i) Find the solution of $y'' - y' - 2y = 0$ that satisfies $y(0) = 1$ and is bounded as $x \to \infty$.

(ii) Solve the related difference equation

$$(y_{n+1} - 2y_n + y_{n-1}) - \frac{1}{2}h\,(y_{n+1} - y_{n-1}) - 2h^2 y_n = 0\,,$$

and show that if $0 < h \ll 1$ the solution that satisfies $y_0 = 1$ and for which $y_n$ is bounded as $n \to \infty$ is approximately $y_n = (1 - h + h^2/2)^n$. Explain the relation with the solution of Part (i).

**5**. Show that

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) \equiv \frac{1}{r}\frac{d^2}{dr^2}(rT)$$

and hence solve the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = k^2 T\,,\ \text{for } r \neq 0$$

subject to the conditions that $\lim_{r\to 0} T(r)$ is finite and $T(1) = 1$.

**6**. Given the solution $y_1(x)$, find a second solution of the following equations:

(i) $x(x+1)y'' + (x-1)y' - y = 0\,,$ with $y_1(x) = (x+1)^{-1}$ ;

(ii) $xy'' - y' - 4x^3 y = 0\,,$ with $y_1(x) = e^{x^2}$ .

$*$**7**. The $n$ functions $y_j(x)$ $(1 \leq j \leq n)$ are independent solutions of the equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = 0\,.$$

Let $\mathbf{W}$ be the $n \times n$ matrix whose $i, j$ element $W_{ij}$ is $y_j^{(i-1)}(x)$ (so that the determinant $|\mathbf{W}| = W$, the Wronskian). Find a matrix $\mathbf{A}$, which does not explicitly involve the $y_j$, such that

$$\mathbf{W}' = \mathbf{A}\mathbf{W}\,,$$

where $\mathbf{W}'$ is the matrix whose elements are given by $(\mathbf{W}')_{ij} = W'_{ij}$. Using the identity

$$(\ln|\mathbf{W}|)' = \mathrm{Tr}\left(\mathbf{W}'\mathbf{W}^{-1}\right) ,$$

express $W$ in terms of $p_1(x)$.

**8**. Let $y(x)$ satisfy the inhomogeneous equation

$$y'' - 2x^{-1}y' + 2x^{-2}y = f(x)\,. \qquad (*)$$

Set

$$\begin{pmatrix} y \\ y' \end{pmatrix} = u(x)\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} + v(x)\begin{pmatrix} y_2 \\ y_2' \end{pmatrix} ,$$

where $y_1(x)$ and $y_2(x)$ are two independent solutions of $(*)$ when $f(x) = 0$, and $u(x)$ and $v(x)$ are unknown functions. Obtain first-order differential equations for $u(x)$ and $v(x)$, and hence find the most general solution of $(*)$ in the case $f(x) = x\sin x$. Are the functions $u(x)$ and $v(x)$ completely determined by this procedure?

**9**. A large oil tanker of mass $M$ floats on the sea of density $\rho$. Suppose the tanker is given a small downward displacement $z$. The upward force is equal to the weight of water displaced (Archimedes' Principle). If the cross-sectional area $A$ of the tanker at the water surface is constant, show that this upward force is $g\rho Az$, and hence that

$$\ddot{z} + \frac{g\rho A}{M}z = 0\,.$$

Suppose now that a mouse exercises on the deck of the tanker producing a vertical force $f\sin\omega t$, where $\omega = (g\rho A/M)^{1/2}$. Show that the tanker will eventually sink. In practice, as the vertical motion of the tanker increases, waves will be generated. Suppose they produce an additional damping force $2k\dot{z}$. Discuss the motion for a range of values of $k$.

**10**. Find and sketch the solution of

$$\ddot{y} + y = H(t-\pi) - H(t-2\pi)\,,$$

where $H$ is the Heaviside step function, subject to the initial conditions

$$y(0) = \dot{y}(0) = 0$$

and with $y(t)$ and $\dot{y}(t)$ continuous at $t = \pi$ and $t = 2\pi$.

**11**. Solve

$$y'' - 4y = \delta(x-a)\,,$$

where $\delta$ is the Dirac delta function, subject to the condition that $y(x)$ is continuous at $x = a$ and boundary conditions that $y$ is bounded as $|x| \to \infty$. Sketch the solution.

**12**. Solve

$$\ddot{y} + 2\dot{y} + 5y = 2\delta(t)\,,$$

where $\delta$ is the Dirac delta function, given that $y = 0$ for $t < 0$. Give an example of a physical system that this equation and boundary condition describes.

$*$**13**. Show that, for suitably chosen $P(x)$, the transformation $y(x) = P(x)v(x)$ reduces the equation

$$y'' + p(x)y' + q(x)y = 0$$

to the form

$$v'' + J(x)v = 0\,. \qquad (*)$$

The sequence of functions $v_n(x)$ is defined, for a given function $J(t)$ and in a given range $0 \le x \le R$, by $v_0(x) = a + bx$ and

$$v_n(x) = \int_0^x (t - x)J(t)v_{n-1}(t)\,dt \qquad (n \ge 1)\,.$$

Show that $v_n''(x) + J(x)v_{n-1} = 0$ $(n \ge 1)$ and deduce that $v(x) = \sum_{n=0}^{\infty} v_n(x)$ satisfies $(*)$ with the initial conditions $v(0) = a,\ v'(0) = b$. [You may assume that the sum that defines $v(x)$ converges sufficiently nicely to allow term-by-term differentiation. In fact, you can show by induction that if $|J(x)| < m$ and $|v_0(x)| < A$ for the range of $x$ under consideration, then $|v_n(x)| \le Am^n x^{2n}/(2n)!$ – try it! Convergence is therefore exponentially fast.]

What do these results tell us about the existence problem for general second-order linear equations with given initial conditions?

$*$**14**. This question concerns the expanding universe. Einstein's equations for a flat, isotropic and homogeneous universe can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}\,, \qquad H \equiv \left(\frac{\dot{a}}{a}\right) = \left(\frac{8\pi}{3}\rho + \frac{\Lambda}{3}\right)^{1/2}\,,$$

where $a$ is the scale factor measuring the expansion of the universe ($\dot{a} > 0$), $\rho$ and $p$ are, respectively, the time-dependent energy density and pressure of matter, $\Lambda$ is the (non-negative) cosmological constant and $H > 0$ is the Hubble parameter. Use these equations to establish the following: if $\Lambda = 0$ (or otherwise negligible) and $\rho + 3p > 0$, the acceleration $\ddot{a} < 0$ and the graph of $a(t)$ must be concave downward implying that $a = 0$ at a finite time in the past (the big bang). Using the tangent of the graph at the present time, $t = t_0$, show that the age of the universe is bounded above by $H^{-1}(t_0)$.

Consider the physical situations of a matter-dominated universe ($\Lambda = 0$ and $p = 0$) and a radiation-dominated universe ($\Lambda = 0$ and $p = \rho/3$). In each case, reduce the two equations above to one single second-order differential equation for $a$ that is homogeneous in $t$ (invariant under $t \to \lambda t$) and then show that there is a solution of the type $a = t^{\alpha}$. Determine the value of $\alpha$ for each case and verify that $\ddot{a} < 0$.

Now consider a $\Lambda$-dominated expanding universe ($\rho = 0$ and $p = 0$). Show that $a \propto e^{\beta t}$ and determine the value of $\beta$. Verify that it corresponds to an accelerated universe ($\ddot{a} > 0$). This could describe the universe today and/or a very early period of exponential expansion known as inflation.

*Comments and corrections may be sent by email to a.d.challinor@ast.cam.ac.uk*