The starred questions are intended as extras: do them if you have time, but not at the expense of questions on later sheets.

1. Find the general solutions of
   (i) \( y'' + 5y' + 6y = e^{3x} \),
   (ii) \( y'' + 9y = \cos 3x \),
   (iii) \( y'' - 2y' + y = (x - 1)e^x \).

2. The function \( y(x) \) satisfies the linear equation
   \[ y'' + p(x)y' + q(x)y = 0. \]
   The Wronskian \( W(x) \) of two independent solutions, denoted \( y_1(x) \) and \( y_2(x) \), is defined to be
   \[ W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \]
   Let \( y_1(x) \) be given. Use the Wronskian to determine a first-order inhomogeneous differential equation for \( y_2(x) \). Hence, show that
   \[ y_2(x) = y_1(x) \int_{x_0}^{x} \frac{W(t)}{y_1(t)^2} \, dt. \] (*)
   Show that \( W(x) \) satisfies
   \[ \frac{dW}{dx} + p(x)W = 0. \]
   Verify that \( y_1(x) = 1 - x \) is a solution of
   \[ xy'' - (1 - x^2)y' - (1 + x)y = 0. \] (†)
   Hence, using (*) with \( x_0 = 0 \) and expanding the integrand in powers of \( t \) to order \( t^3 \), find the first three non-zero terms in the power series expansion for a solution, \( y_2 \), of (†) that is independent of \( y_1 \) and satisfies \( y_2(0) = 0, y_2''(0) = 1 \).

3. Find the general solutions of
   (i) \( y_{n+2} + y_{n+1} - 6y_n = n^2 \),
   (ii) \( y_{n+2} - 3y_{n+1} + 2y_n = n \),
   (iii) \( y_{n+2} - 4y_{n+1} + 4y_n = a^n \), where \( a \neq 2 \). By expressing \( a^n \) as a Taylor series about \( a = 2 \), find the general solution in the case \( a = 2 \).
4. (i) Find the solution of \( y'' - y' - 2y = 0 \) that satisfies \( y(0) = 1 \) and is bounded as \( x \to \infty \).

(ii) Solve the related difference equation

\[
(y_{n+1} - 2y_n + y_{n-1}) - \frac{1}{2}h(y_{n+1} - y_{n-1}) - 2h^2y_n = 0,
\]

and show that if \( 0 < h \ll 1 \) the solution that satisfies \( y_0 = 1 \) and for which \( y_n \) is bounded as \( n \to \infty \) is approximately \( y_n = (1 - h + \frac{1}{2}h^2)^n \). Explain the relation with the solution of (i).

5. Show that

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) \equiv \frac{1}{r} \frac{d^2}{dr^2} (r T)
\]

and hence solve the equation

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = k^2 T, \text{ for } r \neq 0
\]

subject to the conditions that \( \lim_{r \to 0} T(r) \) is finite and \( T(1) = 1 \).

6. Given the solution \( y_1(x) \), find a second solution of the following equations:

(i) \( x(x + 1)y'' + (x - 1)y' - y = 0, \; y_1(x) = (x + 1)^{-1} \);

(ii) \( xy'' - y' - 4x^3y = 0, \; y_1(x) = e^{x^2} \).

*7. The \( n \) functions \( y_j(x) \) (\( 1 \leq j \leq n \)) are independent solutions of the equation

\[
y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = 0.
\]

Let \( W \) be the \( n \times n \) matrix whose \( i, j \) element \( W_{ij} \) is \( y_j^{(i-1)}(x) \) (so that \( \det W = W \), the Wronskian). Find a matrix \( A \), which does not explicitly involve the \( y_j \) such that

\[
W' = A W
\]

where \( W' \) is the matrix whose elements are given by \( (W')_{ij} = W_{ij}' \). Using the identity

\[
(\ln \det W)' = \text{trace} (W'W^{-1}),
\]

express \( W \) in terms of \( p_1(x) \). [You can prove this identity by writing \( W = PDP^{-1} \) where \( D \) is in Jordan normal form (which is upper triangular) and using trace \( ABC = \text{trace} BCA \).]

8. Let \( y(x) \) satisfy the inhomogeneous equation

\[
y'' - 2x^{-1}y' + 2x^{-2}y = f(x).
\]

Set

\[
\begin{pmatrix}
\begin{array}{c}
y \\
y'
\end{array}
\end{pmatrix} = u(x) \begin{pmatrix}
\begin{array}{c}
y_1 \\
y_1'
\end{array}
\end{pmatrix} + v(x) \begin{pmatrix}
\begin{array}{c}
y_2 \\
y_2'
\end{array}
\end{pmatrix},
\]
where \( y_1(x) \) and \( y_2(x) \) are two independent solutions of (\(*\)) when \( f(x) = 0 \), and \( u(x) \) and \( v(x) \) are unknown functions. Obtain first-order differential equations for \( u(x) \) and \( v(x) \), and hence find the most general solution of (\(*\)) in the case \( f(x) = x \sin x \). Are the functions \( u(x) \) and \( v(x) \) completely determined by this procedure?

9. A large oil tanker of mass \( W \) floats on the sea of density \( \rho \). Suppose the tanker is given a small downward displacement \( z \). The upward force is equal to the weight of water displaced (Archimedes’ Principle). If the cross-sectional area \( A \) of the tanker at the water surface is constant, show that this upward force is \( g \rho A z \), and hence that

\[
\ddot{z} + \frac{g \rho A}{W} z = 0 .
\]

Suppose now that a mouse exercises on the deck of the tanker producing a vertical force \( m \sin \omega t \), where \( \omega = (g \rho A/W)^{1/2} \). Show that the tanker will eventually sink. In practice, as the vertical motion of the tanker increases, waves will be generated. Suppose they produce an additional damping \( 2k \dot{z} \). Discuss the motion for a range of values of \( k \).

10. Find and sketch the solution of

\[
\ddot{y} + y = H(t - \pi) - H(t - 2\pi),
\]

where \( H \) is the Heaviside step function, subject to

\[
y(0) = \dot{y}(0) = 0 ,
\]

and with \( y(t) \) and \( \dot{y}(t) \) continuous at \( t = \pi, 2\pi \).

11. Solve

\[
y'' - 4y = \delta(x - a),
\]

where \( \delta \) is the Dirac delta function, subject to the condition that \( y(x) \) is continuous at \( x = a \) and boundary conditions that \( y \) is bounded as \( |x| \to \infty \). Sketch the solution.

12. Solve

\[
\ddot{y} + 2\dot{y} + 5y = 2\delta(t),
\]

where \( \delta \) is the Dirac delta function, given that \( y = 0 \) for \( t < 0 \). Give an example of a physical system that this describes.
13. Show that, for suitably chosen $P(x)$, the transformation $y(x) = P(x)v(x)$ reduces the equation

$$y'' + p(x)y' + q(x)y = 0$$

to the form

$$v'' + J(x)v = 0.$$  

(†)

The sequence of functions $v_n(x)$ is defined, for a given function $J(t)$ and in a given range $0 \leq x \leq R$, by $v_0(x) = a + bx$ and

$$v_n(x) = \int_0^x (t - x)J(t)v_{n-1}(t)dt. \quad (n \geq 1).$$

Show that $v_n''(x) + J(x)v_{n-1} = 0$ ($n \geq 1$) and deduce that $v(x) = \sum_{n=0}^\infty v_n(x)$ satisfies (†) with the initial conditions $v(0) = a$, $v'(0) = b$.

[N.B. You may assume that the sum which defines $v(x)$ converges sufficiently nicely to allow term-by-term differentiation. In fact, you can show by induction that if $|J(x)| < m$ and $|v_0(x)| < A$ for the range of $x$ under consideration, then $|v_n(x)| \leq Am^nx^{2n}/(2n)!$ – try it! Convergence is therefore exponentially fast.]

What does this tell us about the existence problem for general second-order linear equations with given initial conditions?

14 The expanding universe. Einstein’s equations for a flat isotropic and homogeneous universe can be written as:

$$\ddot{a}/a = -4\pi\frac{8\pi}{3} \rho + \frac{\Lambda}{3}$$

$$H \equiv \left(\frac{\dot{a}}{a}\right) = \left(\frac{8\pi}{3} \rho + \frac{\Lambda}{3}\right)^{1/2},$$

where $a$ is the scale factor measuring the expansion of the universe ($\dot{a} > 0$), $\rho$ and $p$ are the time-dependent energy density and pressure of matter, $\Lambda$ is the cosmological constant and $H > 0$ the Hubble parameter. Use these equations to establish the following: If $\Lambda \sim 0$ and $\rho + 3p > 0$ the acceleration $\ddot{a} < 0$ and the graph of $a(t)$ must be concave downward implying that at a finite time $a$ must reach $a = 0$ (the big bang). Using the tangent of the graph at present time $t = t_0$ show that the age of the universe is bounded by $t_0 < H^{-1}(t_0)$.

Consider the physical situations of a matter dominated universe ($\Lambda, p \sim 0$) and a radiation dominated universe ($\Lambda \sim 0, p = \rho/3$). In each case, reduce the two equations above to one single differential equation for $a$ which is homogeneous in $t$ (invariant under $t \rightarrow \lambda t$) and then show that there is a solution of the type $a = t^\alpha$. Determine the value of $\alpha$ for each case and verify that $\ddot{a} < 0$. Now consider a $\Lambda$ dominated universe ($\rho, p << \Lambda$), solve the differential equation for $a(t)$ and show that it corresponds to an accelerated universe ($\ddot{a} > 0$) for $\Lambda > 0$. This could describe the universe today and/or a very early period of exponential expansion known as inflation.

Comments and corrections may be sent by email to J.R.Taylor@damtp.cam.ac.uk