Mathematical Tripos Part IA **Michaelmas Term 2024**

Differential Equations **Prof. Anthony Challinor**

Examples Sheet 4

The starred question is intended as an extra: do it only if you have time.

1. Find two independent series solutions about $x = 0$ of

$$4xy'' + 2(1-x)y' - y = 0\,.$$

2. Find the two independent series solutions about $x = 0$ of

$$y'' - 2xy' + \lambda y = 0$$

for a constant λ. Show that for $\lambda = 2n$, with n a positive integer, one of the solutions is a polynomial of degree n. These are the Hermite polynomials relevant for the solution of the simple harmonic oscillator in quantum mechanics.

3. What is the nature of the point $x = 0$ with respect to the differential equation

$$x^2y'' - xy' + (1-\gamma x)y = 0\,?$$

Find a series solution about $x = 0$ for $\gamma \neq 0$ and write down the form of a second, independent solution. Find two independent solutions of the equation for $\gamma = 0$.

4. Bessel's equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0\,.$$

For $\nu = 0$, find a solution in the form of a power series about $x = 0$.

For $\nu = 1/2$, find two independent series solutions about $x = 0$. Perform also the change of variables $y(x) = z(x)/\sqrt{x}$ to simplify the equation, solve for $z(x)$ and compare with the series result.

5. Find the positions and nature of each of the stationary points of

$$f(x,y) = x^3 + 3xy^2 - 3x\,,$$

and draw a rough sketch of the contours of f.

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6. Find the positions of each of the stationary points of

$$f(x,y) = \sin\left(\frac{x-y}{2}\right)\sin y$$

in $0 < x < 2\pi$, $0 < y < 2\pi$. By using this information and identifying the zero contours of f, sketch the contours of f and identify the nature of the stationary points.

7. For the function $f(x,y,z) = x^2 + y^2 + z^3 - 3z$, find $\boldsymbol{\nabla} f$.

(i) What is the rate of change of $f(x,y,z)$ in the outward normal direction for the cylinder $x^2 + y^2 = 25$ at the point $(3,-4,4)$?

(ii) At which points does $\boldsymbol{\nabla} f$ have no component in the z-direction?

(iii) Find and classify the stationary points of f.

(iv) Sketch the contours of f and add to the sketch a few arrows showing the directions of $\boldsymbol{\nabla} f$.

8. Use matrix methods to solve

$$y' = y - 3z - 6e^x\,, \qquad z' = y + 5z$$

for $y(x)$ and $z(x)$ subject to initial conditions $y(0) = 1$ and $z(0) = 0$.

9. Consider the linear system

$$\dot{\mathbf{x}}(t) + \mathbf{P}\mathbf{x}(t) = \mathbf{z}(t)\,,$$

where $\mathbf{x}(t)$ and $\mathbf{z}(t)$ are 2D vectors, $\mathbf{P}$ is a real constant 2×2 matrix and $\mathbf{z}(t)$ is a given input. Show that free motion (i.e., $\mathbf{z}(t) = 0$) is purely oscillatory (i.e., no growth or decay) if and only if $\mathrm{Tr}(\mathbf{P}) = 0$ and $|\mathbf{P}| > 0$. [The trace of a square matrix is the sum of its diagonal elements.]

Consider

$$\dot{x} + x - y = \cos 2t\,, \qquad \dot{y} + 5x - y = \cos 2t + 2a\sin 2t$$

for various values of the real constant a. For what value(s) of a is there resonance?

[*Hint: it might help to write the forcing terms in the form* $\mathrm{Re}(Ae^{2it})$.]

10. Show that the system

$$\dot{x} = e^{x+y} - y\,,$$
$$\dot{y} = -x + xy$$

has only one fixed (equilibrium) point. Find the linearized system about this point and discuss its stability. Draw the phase portrait near the fixed point.

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11. Use matrix methods to find the general solution of the equations

$$\dot{x} = 3x + 2y\,, \qquad \dot{y} = -5x - 3y\,. \tag{$\dagger$}$$

Sketch the phase-space trajectories in the vicinity of the origin.

$*$Show that the set of equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{x}(t)$ is a column vector and $\mathbf{A}$ is an $n \times n$ matrix with constant elements, has solutions of the form $\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}_0$, where $\mathbf{x}_0$ is a constant vector and

$$\exp(\mathbf{A}t) \equiv \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots\,.$$

Use this method to solve equations ($\dagger$).

Would you expect this method to work if the elements of $\mathbf{A}$ are not constant?

12. Carnivorous hunters of population h prey on vegetarians of population p. In the absence of hunters the prey will increase in number until their population is limited by the availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$\dot{p} = p(1-p) - ph\,, \qquad \dot{h} = \frac{h}{8}\left(\frac{p}{b} - 1\right)\,, \tag{$*$}$$

where b is a positive constant, and $h(t)$ and $p(t)$ are non-negative functions of time t. In the two cases $0 < b < 1/2$ and $b > 1$ determine the location and the stability properties of the equilibrium points of ($*$). In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

13. Consider the change of variables

$$x = e^{-s}\sin t\,, \qquad y = e^{-s}\cos t\,,$$

such that $u(x,y) = v(s,t)$.

(i) Use the chain rule to express $\partial v/\partial s$ and $\partial v/\partial t$ in terms of x, y, $\partial u/\partial x$ and $\partial u/\partial y$.

(ii) Find, similarly, an expression for $\partial^2 v/\partial t^2$.

(iii) Hence transform the equation

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = 0$$

into a partial differential equation for $v(s,t)$.

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14. Solve

$$\frac{\partial y}{\partial t} - 2\frac{\partial y}{\partial x} + y = 0$$

for $y(x,t)$ given $y(x,0) = e^{x^2}$. [*Hint: consider paths in the x–t plane with $x = x_0 - 2t$, where the constants x_0 label the paths.*]

15. The function $\theta(x,t)$ obeys the diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}\,.$$

Find, by substitution, solutions of the form

$$\theta(x,t) = f(t)\exp[-(x+a)^2/(4t+4b)]\,,$$

where a and b are arbitrary constants and the function f is to be determined.

Hence find a solution that satisfies the initial condition

$$\theta(x,0) = \exp[-(x-2)^2] - \exp[-(x+2)^2]$$

and sketch its behaviour for $t \geq 0$.

16. Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (*)$$

for $u(x,y)$ by making a change of variables as follows. Define new variables

$$\xi = x - y\,, \qquad \eta = x\,,$$

and evaluate the partial derivatives of x and y with respect to ξ and η. Writing $v(\xi,\eta) = u(x,y)$, use these derivatives and the chain rule to show that

$$\frac{\partial v}{\partial \eta} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\,,$$

and that the equation

$$\frac{\partial^2 v}{\partial \eta^2} = 0$$

is equivalent to $(*)$.

Deduce that the most general solution of $(*)$ is

$$u(x,y) = f(x-y) + xg(x-y)\,,$$

where f and g are arbitrary functions.

Solve $(*)$ completely given that $u(0,y) = 0$ for all y, whilst $u(x,1) = x^2$ for all x.

Comments and corrections may be sent by email to a.d.challinor@ast.cam.ac.uk.

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