Vector Calculus: Example Sheet 1

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We will have covered the necessary material to do attempt all these questions by the end of lecture 7.

1. Sketch the curve in the plane given parametrically by
   \[ \mathbf{x}(t) = (a \cos^3 t, a \sin^3 t), \quad 0 \leq t \leq 2\pi. \]
   Calculate \( \dot{\mathbf{x}}(t) \) at each point and hence find its total length.

2. A circular helix is given by
   \[ \mathbf{x}(t) = (a \cos t, a \sin t, ct), \quad t \in \mathbb{R}. \]
   Calculate the tangent \( \mathbf{t} \), curvature \( \kappa \), principal normal \( \mathbf{n} \), binormal \( \mathbf{b} \) and torsion \( \tau \).
   Give a sketch of the curve indicating the directions of the vectors \( \{\mathbf{t}, \mathbf{n}, \mathbf{b}\} \).

3. Show that a planar curve \( \mathbf{x}(t) = (x(t), y(t), 0) \) has curvature
   \[ \kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \]
   Find the minimum and maximum values of the curvature on the ellipse
   \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

4. Evaluate explicitly each of the line integrals
   \[ \int (x \, dx + y \, dy + z \, dz), \quad \int (y \, dx + x \, dy + dz), \quad \int (y \, dx - x \, dy + e^{x+y} \, dz) \]
   along (i) the straight line path from the origin to \((1, 1, 1)\), and (ii) the parabolic path given parametrically by \((x, y, z) = (t, t, t^2)\) from \(t = 0\) to \(t = 1\). For which of these integrals do the two paths give the same results, and why?
5. Consider the vector fields \( \mathbf{F}(x) = (3x^2yz^2, 2x^3yz, x^3z^2) \) and \( \mathbf{G}(x) = (3x^2y^2z, 2x^3yz, x^3y^2) \). Compute the line integrals \( \int \mathbf{F} \cdot d\mathbf{x} \) and \( \int \mathbf{G} \cdot d\mathbf{x} \) along the following paths, each of which consist of straight line segments joining the specified points:

(i) \((0,0,0) \rightarrow (1,1,1)\),
(ii) \((0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1)\),
(iii) \((0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)\).

Are either of the differentials \( \mathbf{F} \cdot d\mathbf{x} \) or \( \mathbf{G} \cdot d\mathbf{x} \) exact?

6. A curve \( C \) is given parametrically by

\[
x(t) = (\cos(\sin nt) \cos t, \cos(\sin nt) \sin t, \sin(\sin nt)) \quad 0 \leq t \leq 2\pi,
\]

where \( n \) is some fixed integer. Sketch the curve. Show that

\[
\oint_C \mathbf{F} \cdot d\mathbf{x} = 2\pi,
\]

where \( \mathbf{F}(x) = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right) \)

and \( C \) is traversed in the direction of increasing \( t \). Can \( \mathbf{F}(x) \) be written as the gradient of a scalar function? Comment on your results.

[Hint: when sketching the curve, you may find it helpful to use spherical polar coordinates, defined by \( x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi \) and \( z = r \cos \theta \) with \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \).]

7. Use the substitution \( x = r \cos \phi, y = \frac{1}{2} r \sin \phi \), to evaluate

\[
\int_D \frac{x^2}{x^2+4y^2} \, dA
\]

where \( D \) is the region between the two ellipses \( x^2 + 4y^2 = 1, x^2 + 4y^2 = 4 \).

8. A closed curve \( C \) in the \( z = 0 \) plane consists of the arc of the parabola \( y^2 = 4ax \) \((a > 0)\) between the points \((a, \pm 2a)\) and the straight line joining \((a, \mp 2a)\). The area inclosed by \( C \) is \( D \). Show, by calculating the integrals explicitly, that

\[
\oint_C (x^2 y \, dx + xy^2 \, dy) = \int_D (y^2 - x^2) \, dA = \frac{104}{105} a^4
\]
9. The region $D$ is bounded by the segments $x = 0, \ 0 \leq y \leq 1; \ y = 0, \ 0 \leq x \leq 1; \ y = 1, \ 0 \leq x \leq \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the $(x, y)$-plane from the $(u, v)$-plane defined by the transformation $x = u^2 - v^2, \ y = 2uv$. Sketch $D$ and also the two regions in the $(u, v)$-plane which are mapped into it. Hence evaluate

$$\int_D \frac{dA}{(x^2 + y^2)^{1/2}}$$

10. Compute the volume of a cone of height $h$ and radius $a$ using (a) cylindrical polars, (b) spherical polars.

11. By using a suitable change of variables, calculate the volume within an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

12*. Compute the volume of the region $V$ defined by the intersection of the three cylinders

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, \ y^2 + z^2 \leq 1, \ z^2 + x^2 \leq 1\}.$$

[Warning: The sole purpose of this question is to show you that volume integrals can be arbitrarily hard. Only attempt if that’s your thing.]