Vector Calculus: Example Sheet 1

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1. Let $\phi(x)$ be a scalar field and $v(x)$ a vector field. Show, using suffix notation, that

$$\nabla \cdot (\phi v) = (\nabla \phi) \cdot v + \phi (\nabla \cdot v), \quad \nabla \times (\phi v) = (\nabla \phi) \times v + \phi (\nabla \times v).$$

Evaluate the divergence and curl of the following:

$$rx, \quad a(x \cdot b), \quad a \times x, \quad x/r^3,$$

where $r = |x|$ and $a, b$ are constant vectors.

2. For vector fields $u(x)$ and $v(x)$, use suffix notation to show that,

$$\nabla \times (u \times v) = u(\nabla \cdot v) + (v \cdot \nabla)u - v(\nabla \cdot u) - (u \cdot \nabla)v,$$

Show also that

$$(u \cdot \nabla)u = \nabla \left( \frac{1}{2} |u|^2 \right) - u \times (\nabla \times u).$$

3. Verify directly that the vector field

$$u(x) = (e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0)$$

is irrotational and express is as the gradient of a scalar field $\phi$. Check that $u$ is solenoidal and show that it can be written as the curl of the vector field $v = (0, 0, \psi)$, for some function $\psi$.

4. Check that the following vector field is irrotational

$$F = (3x^2 \tan z - y^2 e^{-xy^2} \sin y, (\cos y - 2xy \sin y) e^{-xy^2}, x^3 \sec^2 z)$$

Find the most general scalar potential $\phi(x)$ such that $F = \nabla \phi$.

5. Sketch the curve in the plane given parametrically by

$$x(t) = (a \cos^3 t, a \sin^3 t), \quad 0 \leq t \leq 2\pi.$$

Calculate $\dot{x}(t)$ at each point and hence find its total length.
6. A circular helix is given by
\[ x(t) = (a \cos t, a \sin t, ct), \quad t \in \mathbb{R}. \]
Calculate the tangent \( t \), curvature \( \kappa \), principal normal \( n \), binormal \( b \) and torsion \( \tau \).
Give a sketch of the curve indicating the directions of the vectors \( \{t, n, b\} \).

7. Show that a planar curve \( x(t) = (x(t), y(t), 0) \) has curvature
\[ \kappa(t) = \frac{\lvert \dot{x} \ddot{y} - \ddot{x} \dot{y} \rvert}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \]
Find the minimum and maximum values of the curvature on the ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

8. The vector field \( B(x) \) is everywhere parallel to the normals of a family of surfaces \( f(x) = \text{constant} \). Show that \( B \cdot (\nabla \times B) = 0 \).

The tangent vector at each point on a curve is parallel to a non-vanishing vector field \( H(x) \). Show that the curvature of the curve is given by \( \kappa = \lvert H \times (H \cdot \nabla)H \rvert / |H^3| \).

9. Evaluate explicitly each of the line integrals
\[ \int (x \, dx + y \, dy + z \, dz), \quad \int (y \, dx + x \, dy + dz), \quad \int (y \, dx - x \, dy + e^{x+y} \, dz) \]
along (i) the straight line path from the origin to \((1, 1, 1)\), and (ii) the parabolic path given parametrically by \((x, y, z) = (t, t, t^2)\) from \( t = 0 \) to \( t = 1 \). For which of these integrals do the two paths give the same results, and why?

10. Consider the vector fields \( F(x) = (3x^2yz^2, 2x^3yz, x^3z^2) \) and \( G(x) = (3x^2y^2z, 2x^3yz, x^3y^2) \).
Compute the line integrals \( \int F \cdot dx \) and \( \int G \cdot dx \) along the following paths, each of which consist of straight line segments joining the specified points:
   (i) \((0,0,0) \to (1,1,1)\),
   (ii) \((0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)\),
   (iii) \((0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)\).
Are either of the differentials \( F \cdot dx \) or \( G \cdot dx \) exact?
11. A curve $C$ is given parametrically by

$$x(t) = (\cos(\sin(nt)) \cos t, \cos(\sin(nt)) \sin t, \sin(\sin(nt))) \quad 0 \leq t \leq 2\pi,$$

where $n$ is some fixed integer. Sketch the curve. Show that

$$\oint_C F \cdot dx = 2\pi,$$

where $F(x) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ and $C$ is traversed in the direction of increasing $t$. Can $F(x)$ be written as the gradient of a scalar function? Comment on your results.

[Hint: when sketching the curve, you may find it helpful to use spherical polar coordinates, defined by $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$.)]

12. Use the substitution $x = r \cos \phi$, $y = \frac{1}{2}r \sin \phi$, to evaluate

$$\int_D \frac{x^2}{x^2+4y^2} \, dA$$

where $D$ is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

13. A closed curve $C$ in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ ($a > 0$) between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area inclosed by $C$ is $D$. Show, by calculating the integrals explicitly, that

$$\oint_C (x^2y \, dx + xy^2 \, dy) = \int_D (y^2 - x^2) \, dA = \frac{104}{105}a^4$$

14. The region $D$ is bounded by the segments $x = 0$, $0 \leq y \leq 1$; $y = 0$, $0 \leq x \leq 1$; $y = 1$, $0 \leq x \leq \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the $(x, y)$-plane from the $(u, v)$-plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch $D$ and also the two regions in the $(u, v)$-plane which are mapped into it. Hence evaluate

$$\int_D \frac{dA}{(x^2+y^2)^{1/2}}$$

15*. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^3$ is divergence free, i.e. $\nabla \cdot F = 0$. Show that $F = \nabla \times A$ where

$$A(x) = \int_0^1 F(tx) \times (tx) \, dt.$$

What goes wrong with this formula if $F$ is not defined on all of $\mathbb{R}^3$?