1. Compute the volume of a cone of height \( h \) and radius \( a \) using (a) cylindrical polars, (b) spherical polars.
2. By using a suitable change of variables, calculate the volume within an ellipsoid
   \[
   \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.
   \]
3. Let \((u, v, w)\) be a set of orthogonal curvilinear coordinates for \( \mathbb{R}^3 \). Show that
   \[
   dV = h_u h_v h_w \, du \, dv \, dw.
   \]
   Confirm that \( dV = \rho d\rho d\phi dz \) and \( dV = r^2 \sin \theta \, d\rho \, d\theta \, d\phi \) an cylindrical and spherical polars respectively.
4. Let \( f = f(x) \) be a scalar field and \( \mathbf{v} = \mathbf{v}(x) \) a vector field. Show, using suffix notation, that
   \[
   \nabla \cdot (f \mathbf{v}) = (\nabla f) \cdot \mathbf{v} + f(\nabla \cdot \mathbf{v}), \quad \nabla \times (f \mathbf{v}) = (\nabla f) \times \mathbf{v} + f(\nabla \times \mathbf{v}).
   \]
   Evaluate the divergence and curl of the following:
   \[
   r \mathbf{x}, \quad \mathbf{a} \times \mathbf{b}, \quad \mathbf{a} \times \mathbf{x}, \quad \mathbf{x}/r^3,
   \]
   where \( r = |\mathbf{x}| \) and \( \mathbf{a}, \mathbf{b} \) are constant vectors.
5. The vector field \( \mathbf{A} = \mathbf{A}(x) \) is, in Cartesian, cylindrical and spherical polar coordinates respectively,
   \[
   \mathbf{A}(x) = -\frac{1}{2} y \mathbf{e}_x + \frac{1}{2} x \mathbf{e}_y = \frac{1}{2} \rho \mathbf{e}_\phi = \frac{1}{2} r \sin \theta \mathbf{e}_\phi.
   \]
   Compute the \( \nabla \times \mathbf{A} \) in each different coordinate system and check that your answers agree.
6. Recall that in cylindrical polar coordinates
   \[
   \nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}
   \]
   while all other derivatives are zero. Derive expressions for the \( \nabla \cdot \mathbf{A} \) and \( \nabla \times \mathbf{A} \) where \( \mathbf{A} \) is an arbitrary vector field given in cylindrical polars by \( \mathbf{A} = A_\rho \mathbf{e}_\rho + A_\phi \mathbf{e}_\phi + A_z \mathbf{e}_z \). Also derive an expression for the Laplacian of a scalar function \( \nabla^2 f \) in this coordinate system.
7. Use suffix notation to show that
   \[
   \nabla \times (u \times v) = u(\nabla \cdot v) + (\nabla \cdot u)v - v(\nabla \cdot u) - (u \cdot \nabla)v,
   \]
   for vector fields \( u = u(x) \) and \( v = v(x) \). Show also that \( (u \cdot \nabla)u = \nabla \left( \frac{1}{2} |u|^2 \right) - u \times (\nabla \times u) \).
8. Verify directly that the vector field
   \[
   u(x) = (e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0)
   \]
   is irrotational and express is as the gradient of a scalar field \( \phi \). Check that \( u \) is solenoidal and show that it can be written as the curl of the vector field \( v = (0, 0, \psi) \), for some function \( \psi \).
9. Consider the line integral
   \[
   I = \int_C -x^2 y \, dx + xy^2 \, dy
   \]
   for \( C \) a closed curve traversed anti-clockwise in the \((x, y)\)-plane.
   (i) Evaluate \( I \) when \( C \) is a circle of radius \( R \) centred at the origin. Use Green’s theorem to relate the results for \( R = b \) and \( R = a \) to an area integral over an appropriate region, and calculate the area integral directly.
   (ii) Now suppose \( C \) is the boundary of a square centred at the origin with sides of length \( \ell \). Show that \( I \) is independent of the orientation of the square in the \((x, y)\)-plane.
10. Verify Stokes’ theorem for the hemispherical shell \( S = \{ x^2 + y^2 + z^2 = 1, z \geq 0 \} \), and the vector field 
\( \mathbf{F}(x) = (y, -x, z) \).

11. By applying Stokes’ theorem to the vector field \( a \times \mathbf{F} \) for a constant, or otherwise, show that for a vector field \( \mathbf{F} = \mathbf{F}(x) \)
\[ \oint_C d\mathbf{x} \times \mathbf{F} = \int_S (d\mathbf{S} \times \nabla) \times \mathbf{F} \]
where \( C = \partial S \). Verify this result when \( C \) is the unit square in the \((x, y)\)-plane with opposite vertices at \((0, 0, 0)\) and \((1, 1, 0)\) and \( \mathbf{F}(x) = x \).

12. Let \( S = \{ x : |x| = 1 \} \). For the vector field \( \mathbf{F}(x) = x/r^3 \), where \( r = |x| \), compute the integral 
\[ \int_S \mathbf{F} \cdot d\mathbf{S} \]
Deduce that there does not exist a vector potential for \( \mathbf{F} \). Compute \( \nabla \cdot \mathbf{F} \) and comment on your result.

Additional problems
These questions should not be attempted at the expense of earlier ones.

13. Compute the volume of the region \( V \) defined by the intersection of the three cylinders 
\( V = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, y^2 + z^2 \leq 1, z^2 + x^2 \leq 1 \} \).

14. Let \( \mathbf{F}(x) = x/r^3 \) as in question 12. Consider the vector field 
\( \mathbf{A}(x) = \left( \frac{yz}{(x^2 + y^2)^{3/2}}, \frac{zz}{(x^2 + y^2)^{3/2}}, \frac{xx}{(x^2 + y^2)^{3/2}} \right) \).
Show that \( \nabla \times \mathbf{A} = \mathbf{F} \). Does this contradict the result of question 12? Reconcile these results by applying Stokes’ theorem on the surface \( S = \{ x : |x| = 1, x^2 + y^2 \geq 1 \} \) and taking a suitable limit.
*Hint: you may find cylindrical or spherical polar coordinates helpful for this question.*

15. Suppose \( \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) is a solenoidal vector field, i.e. \( \nabla \cdot \mathbf{F} = 0 \). Show that \( \mathbf{F} = \nabla \times \mathbf{A} \) where 
\[ \mathbf{A}(x) = \int_0^1 \mathbf{F}(tx) \times (tx) \, dt. \]
This is called a homotopy formula. What goes wrong with this formula if \( \mathbf{F} \) is not defined on all of \( \mathbb{R}^3 \)?