Vector Calculus: Example Sheet 2

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1. Obtain the equation of the plane which is tangent to the surface
   \[ z = 3x^2y \sin(\pi x/2) \]
   at the point \( x = y = 1 \).
   
   Take East to be in the direction \((1, 0, 0)\) and North to be \((0, 1, 0)\). In which direction
   will a marble roll if placed on the surface at \( x = 1, y = \frac{1}{2} \)?

2. The vector field \( \mathbf{B}(x) \) is everywhere parallel to the normals of a family of surfaces
   \( f(x) = \text{constant} \). Show that \( \mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0 \).
   
   The tangent vector at each point on a curve is parallel to a non-vanishing vector
   field \( \mathbf{H}(x) \). Show that the curvature of the curve is given by
   \[ \kappa = \left| \mathbf{H} \times (\mathbf{H} \cdot \nabla) \mathbf{H} \right| / |\mathbf{H}|^3 \].

3. Let \( \phi(x) \) be a scalar field and \( \mathbf{v}(x) \) a vector field. Show, using suffix notation, that
   \[ \nabla \cdot (\phi \mathbf{v}) = (\nabla \phi) \cdot \mathbf{v} + \phi(\nabla \cdot \mathbf{v}), \quad \nabla \times (\phi \mathbf{v}) = (\nabla \phi) \times \mathbf{v} + \phi(\nabla \times \mathbf{v}) \].

   Evaluate the divergence and curl of the following:
   \[ r \mathbf{x}, \quad \mathbf{a}(\mathbf{x} \cdot \mathbf{b}), \quad \mathbf{a} \times \mathbf{x}, \quad \mathbf{x}/r^3, \]
   where \( r = |\mathbf{x}| \) and \( \mathbf{a}, \mathbf{b} \) are constant vectors.

4. For vector fields \( \mathbf{u}(x) \) and \( \mathbf{v}(x) \), use suffix notation to show that,
   \[ \nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{v}, \]
   Show also that
   \[ (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u}) \]

5. Verify directly that the vector field
   \[ \mathbf{u}(x) = (e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0) \]
   is irrotational and express is as the gradient of a scalar field \( \phi \). Check that \( \mathbf{u} \) is
   solenoidal and show that it can be written as the curl of the vector field \( \mathbf{v} = (0, 0, \psi) \),
   for some function \( \psi \).
6. Check that the following vector field is irrotational
\[ F = (3x^2 \tan z - y^2 e^{-xy^2} \sin y, (\cos y - 2xy \sin y) e^{-xy^2}, x^3 \sec^2 z) \]
Find the most general scalar potential \( \phi(x) \) such that \( F = \nabla \phi \).

7*. Suppose \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) is divergence free, i.e. \( \nabla \cdot F = 0 \). Show that \( F = \nabla \times A \) where
\[ A(x) = \int_0^1 F(tx) \times (tx) \, dt. \]
What goes wrong with this formula if \( F \) is not defined on all of \( \mathbb{R}^3 \)?

8. Let \((u, v, w)\) be a set of orthogonal curvilinear coordinates for \( \mathbb{R}^3 \). Show that
\[ dV = h_u h_v h_w \, du \, dv \, dw. \]
Confirm that \( dV = \rho \, d\rho \, d\phi \, dz \) and \( dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \) an cylindrical and spherical polars respectively.

9. If \( a \) is constant vector and \( r = |x| \), verify that
\[ \nabla (r^n) = nr^{n-2} x, \quad \nabla (a \cdot x) = a \]
using (i) Cartesian coordinates and suffix notation, (ii) cylindrical polar coordinates, (iii) spherical polar coordinates, and (iv) a first principles, coordinate-independent approach.
[Note: for parts (ii) and (iii) you will need to be careful with the components of \( a \) with respect to each of the relevant bases.]

10. The vector field \( A(x) \) is, in Cartesian, cylindrical and spherical polar coordinates respectively,
\[ A(x) = -\frac{1}{2} y e_x + \frac{1}{2} x e_y = \frac{1}{2} \rho e_\phi = \frac{1}{2} r \sin \theta e_\phi. \]
Compute the \( \nabla \times A \) in each different coordinate system and check that your answers agree.

11. Recall that in cylindrical polar coordinates
\[ \nabla = e_\rho \frac{\partial}{\partial \rho} + e_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + e_z \frac{\partial}{\partial z} \]
and
\[ \frac{\partial e_\rho}{\partial \phi} = e_\phi, \quad \frac{\partial e_\phi}{\partial \phi} = -e_\rho, \]
while all other derivatives are zero. Derive expressions for the \( \nabla \cdot A \) and \( \nabla \times A \) where \( A \) is an arbitrary vector field given in cylindrical polars by \( A = A_\rho e_\rho + A_\phi e_\phi + A_z e_z \). Also derive an expression for the Laplacian of a scalar function \( \nabla^2 f \) in this coordinate system.
12. By applying the divergence theorem to the vector field $a \times A$, where $a$ is an arbitrary constant vector and $A(x)$ is a vector field, show that

$$\int_V \nabla \times A \, dV = \int_S \mathbf{dS} \times A$$

where $S = \partial V$. Verify this result when $V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and $A(x) = (z, 0, 0)$.

13. Let $\mathbf{F}(x) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$ and let $S$ be the open surface

$$1 - z = x^2 + y^2, \quad 0 \leq z \leq 1.$$

Use the divergence theorem and cylindrical polar coordinates to evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$. Verify your result by calculating the area integral directly.

[Hint: you should find that $d\mathbf{S} = (2\rho \cos \phi, 2\rho \sin \phi, 1) \rho \, d\rho \, d\phi$.]

14. For the electric and magnetic fields $\mathbf{E}(x, t)$ and $\mathbf{B}(x, t)$ define the quantities

$$U = \frac{1}{2} \left( \varepsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0} |\mathbf{B}|^2 \right), \quad \mathbf{P} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

Use Maxwell’s equations with $\mathbf{J} = 0$ to establish the conservation law $\partial U/\partial t + \nabla \cdot \mathbf{P} = 0$. If $U(x)$ has the interpretation of the energy density stored in electric and magnetic fields, what is the interpretation of the so-called Poynting vector $\mathbf{P}$?