Vector Calculus: Example Sheet 2

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The material necessary to attempt all questions will have been covered by the end of lecture 13.

1. Obtain the equation of the plane which is tangent to the surface \( z = 3x^2y \sin(\pi x/2) \) at the point \( x = y = 1 \).

   Take East to be in the direction \((1, 0, 0)\) and North to be \((0, 1, 0)\). In which direction will a marble roll if placed on the surface at \( x = 1, y = \frac{1}{2} \)?

2. The vector field \( B(x) \) is everywhere parallel to the normals of a family of surfaces \( f(x) = \text{constant} \). Show that \( B \cdot (\nabla \times B) = 0 \).

   The tangent vector at each point on a curve is parallel to a non-vanishing vector field \( H(x) \). Show that the curvature of the curve is given by \( \kappa = |H \times (H \cdot \nabla)H|/|H^3| \).

3. Let \( \phi(x) \) be a scalar field and \( v(x) \) a vector field. Show, using suffix notation, that

\[
\nabla \cdot (\phi v) = (\nabla \phi) \cdot v + \phi (\nabla \cdot v) , \quad \nabla \times (\phi v) = (\nabla \phi) \times v + \phi (\nabla \times v).
\]

Evaluate the divergence and curl of the following:

\( rx, \ a(x \cdot b), \ a \times x, \ x/r^3, \)

where \( r = |x| \) and \( a, b \) are constant vectors.

4. For vector fields \( u(x) \) and \( v(x) \), use suffix notation to show that,

\[
\nabla \times (u \times v) = u(\nabla \cdot v) + (v \cdot \nabla)u - v(\nabla \cdot u) - (u \cdot \nabla)v.
\]

Show also that

\[
(u \cdot \nabla)u = \nabla \left( \frac{1}{2} |u|^2 \right) - u \times (\nabla \times u)
\]

5. Verify directly that the vector field

\[
u(x) = (e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0)
\]

is irrotational and express is as the gradient of a scalar field \( \phi \). Check that \( u \) is solenoidal and show that it can be written as the curl of the vector field \( v = (0, 0, \psi) \), for some function \( \psi \).
6. Check that the following vector field is irrotational
\[ F = (3x^2 \tan z - y^2 e^{-x^2} \sin y, \cos y - 2xy \sin y) e^{-xy^2}, x^3 \sec^2 z) \]
Find the most general scalar potential \( \phi(x) \) such that \( F = \nabla \phi \).

7*. Suppose \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) is divergence free, i.e. \( \nabla \cdot F = 0 \). Show that \( F = \nabla \times A \) where
\[ A(x) = \int_0^1 F(tx) \times (tx) \, dt. \]
What goes wrong with this formula if \( F \) is not defined on all of \( \mathbb{R}^3 \)?

8. Let \((u, v, w)\) be a set of orthogonal curvilinear coordinates for \( \mathbb{R}^3 \). Show that
\[ dV = h_u h_v h_w \, du \, dv \, dw. \]
Confirm that \( dV = \rho \, d\rho \, d\phi \, dz \) and \( dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \) an cylindrical and spherical polars respectively.

9. If \( a \) is constant vector and \( r = |x| \), verify that
\[ \nabla (r^n) = nr^{n-2}x, \quad \nabla (a \cdot x) = a \]
using (i) Cartesian coordinates and suffix notation, (ii) cylindrical polar coordinates, (iii) spherical polar coordinates, and (iv) a first principles, coordinate-independent approach.

[Note: for parts (ii) and (iii) you will need to be careful with the components of \( a \) with respect to each of the relevant bases.]

10. The vector field \( A(x) \) is, in Cartesian, cylindrical and spherical polar coordinates respectively,
\[ A(x) = -\frac{1}{2}y e_x + \frac{1}{2}x e_y = \frac{1}{2} \rho e_{\phi} = \frac{1}{2} r \sin \theta e_{\phi}. \]
Compute the \( \nabla \times A \) in each different coordinate system and check that your answers agree.

11. Recall that in cylindrical polar coordinates
\[ \nabla = e_\rho \frac{\partial}{\partial \rho} + e_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + e_z \frac{\partial}{\partial z}, \quad \frac{\partial e_\rho}{\partial \phi} = e_\phi, \quad \frac{\partial e_\phi}{\partial \phi} = -e_\rho, \]
while all other derivatives are zero. Derive expressions for the \( \nabla \cdot A \) and \( \nabla \times A \) where \( A \) is an arbitrary vector field given in cylindrical polars by \( A = A_\rho e_\rho + A_\phi e_\phi + A_z e_z \).
Also derive an expression for the Laplacian of a scalar function \( \nabla^2 f \) in this coordinate system.
12. By applying the divergence theorem to the vector field $\mathbf{a} \times \mathbf{A}$, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field, show that

$$\int_V \nabla \times \mathbf{A} \, dV = \int_S \mathbf{dS} \times \mathbf{A}$$

where $S = \partial V$. Verify this result when $V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and $\mathbf{A}(\mathbf{x}) = (z, 0, 0)$.

13. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, x^2 + y^2 + 3z^2)$ and let $S$ be the open surface

$$1 - z = x^2 + y^2, \quad 0 \leq z \leq 1.$$ 

Use the divergence theorem and cylindrical polar coordinates to evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$. Verify your result by calculating the area integral directly.

[Hint: you should find that $d\mathbf{S} = (2\rho \cos \phi, 2\rho \sin \phi, 1) \rho \, d\rho \, d\phi$.]

14. For the electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ define the quantities

$$U = \frac{1}{2} \left( \varepsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0} |\mathbf{B}|^2 \right), \quad \mathbf{P} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$ 

Use Maxwell’s equations with $\mathbf{J} = 0$ to establish the conservation law $\partial U/\partial t + \nabla \cdot \mathbf{P} = 0$.

If $U(\mathbf{x})$ has the interpretation of the energy density stored in electric and magnetic fields, what is the interpretation of the so-called Poynting vector $\mathbf{P}$?