Vector Calculus: Example Sheet 2 Part IA, Lent Term 2025 Dr R. E. Hunt

Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Line Integrals

1. Evaluate explicitly each of the line integrals

$$\int (x \, \mathrm{d}x + y \, \mathrm{d}y + z \, \mathrm{d}z), \quad \int (y \, \mathrm{d}x + x \, \mathrm{d}y + \mathrm{d}z), \quad \int (y \, \mathrm{d}x - x \, \mathrm{d}y + e^{x+y} \, \mathrm{d}z),$$

along (i) the straight line path from the origin to (1, 1, 1), and (ii) the parabolic path given parametrically by $(x, y, z) = (t, t, t^2)$ from t = 0 to t = 1.

For which of these integrals do the two paths give the same results, and why?

- **2.** Consider the two vector fields $\mathbf{F}(\mathbf{x}) = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G}(\mathbf{x}) = (3x^2y^2z, 2x^3yz, x^3y^2)$. Compute the line integrals $\int \mathbf{F} \cdot d\mathbf{x}$ and $\int \mathbf{G} \cdot d\mathbf{x}$ along the following paths, each of which consist of straight line segments joining the specified points:
 - (i) $(0,0,0) \to (1,1,1),$
 - (ii) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1),$
 - (iii) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1).$

Are either of the differentials **F** . dx or **G** . dx exact?

3. A curve *C* is given parametrically by

 $\mathbf{x}(t) = \left(\cos(\sin nt)\cos t, \cos(\sin nt)\sin t, \sin(\sin nt)\right), \quad 0 \le t \le 2\pi,$

where *n* is some fixed integer. Using spherical polar coordinates, or otherwise, sketch the curve. Show that if

$$\mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right)$$

and *C* is traversed in the direction of increasing *t* then $\oint_C \mathbf{F} \cdot d\mathbf{x} = 2\pi$.

Find a scalar function *f* such that $\mathbf{F}(\mathbf{x}) = \nabla f$. Comment on this, given your prior result.

Area Integrals

4. A closed curve \mathscr{C} in the z = 0 plane consists of the arc of the parabola $y^2 = 4ax$ (a > 0) between the points ($a, \pm 2a$), and the straight line joining ($a, \mp 2a$). The area enclosed by \mathscr{C} is \mathscr{D} . Show, by calculating the integrals explicitly, that

$$\int_{\mathscr{C}} (x^2 y \, \mathrm{d}x + x y^2 \, \mathrm{d}y) = \int_{\mathscr{D}} (y^2 - x^2) \, \mathrm{d}A = \frac{104}{105} a^4.$$

5. Use the substitution $x = r \cos \phi$, $y = \frac{1}{2}r \sin \phi$ to show that

$$\int_D \frac{x^2}{x^2 + 4y^2} \,\mathrm{d}A = \frac{3}{4}\pi$$

where *D* is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

6. The region *D* is bounded by the segments $x = 0, 0 \le y \le 1$; $y = 0, 0 \le x \le 1$; $y = 1, 0 \le x \le \frac{3}{4}$; and an arc of the parabola $y^2 = 4(1 - x)$. Consider a mapping into the (x, y)-plane from the (u, v)-plane defined by the transformation $x = u^2 - v^2$, y = 2uv. Sketch *D* and also the two regions in the (u, v)-plane that are mapped into it. Hence evaluate

$$\int_D \frac{\mathrm{d}A}{\left(x^2 + y^2\right)^{1/2}}$$

* 7. Let f(x) be a smooth function such that $\int_{1}^{\infty} \frac{1}{x} f(x) dx$ exists. By considering an appropriate double integral involving f' and changing the order of integration, show that

$$\int_0^\infty \frac{f(bx) - f(ax)}{x} \, \mathrm{d}x = f(0) \log(a/b), \quad a, b > 0.$$

Volume Integrals

- 8. Calculate the volume within the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by using a suitable change of variables.
- **9.** (i) Compute the volume of a cone of height *h* and radius *a* using cylindrical polars.
 - (ii) The apex of the cone is placed at the origin and its axis aligned with the positive *z*-axis so that its base lies in the plane z = h. The density of the cone is then given by $\rho_0 \cos \theta$ in *spherical* polars. Show that its total mass is $\frac{2}{3}\pi\rho_0 h^3(\sqrt{1+a^2/h^2}-1)$.
- **10.** A tetrahedron *V* of uniform density ρ_0 has vertices at (1, 0, 0), (0, 1, 0), (0, 0, 1) and the origin. Find its centre of mass, defined by $\mathbf{X} = M^{-1} \int_V \rho_0 \mathbf{x} \, \mathrm{d}^3 \mathbf{x}$ where *M* is the total mass. [*Answer*: $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.]
- **11.** Let (u, v, w) be a set of orthogonal curvilinear coordinates for \mathbb{R}^3 . Show that a volume element is given by

$$\mathrm{d}V = h_u h_v h_w \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w.$$

Confirm that in cylindrical and spherical polars, $dV = \rho d\rho d\phi dz$ and $dV = r^2 \sin \theta dr d\theta d\phi$ respectively. Draw diagrams to illustrate these two results.

* 12. Let *a*, *b*, *c* > 0. Using the new variables u = x/y, v = xy, w = yz, or otherwise, show that

$$\int_{x=0}^{\infty} \int_{y=0}^{1} \int_{z=0}^{x} x e^{-ax/y - bxy - cyz} \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{2a(a+b)(a+b+c)}$$

* **13.** Compute the volume of the region *V* defined by the intersection of three cylinders as follows:

$$V = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1, y^2 + z^2 \le 1, z^2 + x^2 \le 1 \right\}$$

[Tricky, but fun. Answer: $8(2 - \sqrt{2})$.]

Green's Theorem in the Plane

14. Consider the line integral

$$I = \oint_C (-x^2 y \, \mathrm{d}x + x y^2 \, \mathrm{d}y)$$

where *C* is a closed curve traversed anticlockwise in the (x, y)-plane.

- (i) Evaluate *I* when *C* is a circle of radius *R* centred at the origin. Use Green's theorem to relate the results for R = b and R = a to an area integral over an appropriate region, and calculate the area integral directly.
- (ii) Now suppose that *C* is the boundary of a square centred at the origin with sides of length *l*. Show that *I* is independent of the orientation of the square in the (x, y)-plane.