

Example Sheet 2

1 A circular helix is given by

$$\mathbf{x}(u) = (a \cos u, a \sin u, cu).$$

Calculate the tangent \mathbf{t} , curvature κ , principal normal \mathbf{n} , binormal \mathbf{b} , and torsion τ .

2 Show that a curve in the plane, $\mathbf{r}(t) = (x(t), y(t), 0)$, has curvature

$$\kappa(t) = |\dot{x}\ddot{y} - \dot{y}\ddot{x}| / (\dot{x}^2 + \dot{y}^2)^{3/2}.$$

Find the minimum and maximum curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ ($a > b > 0$).

3 Let $\psi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using suffix notation, that

$$\nabla \cdot (\psi \mathbf{v}) = (\nabla \psi) \cdot \mathbf{v} + \psi \nabla \cdot \mathbf{v}, \quad \nabla \times (\psi \mathbf{v}) = (\nabla \psi) \times \mathbf{v} + \psi \nabla \times \mathbf{v}.$$

Evaluate (using suffix notation where necessary) the divergence and the curl of the following:

$$r \mathbf{x}, \quad \mathbf{a}(\mathbf{x} \cdot \mathbf{b}), \quad \mathbf{a} \times \mathbf{x}, \quad \mathbf{x}/r^3,$$

where $r = |\mathbf{x}|$, and \mathbf{a} and \mathbf{b} are fixed vectors.

4 Use suffix notation to show that

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla) \mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla) \mathbf{v}.$$

for vector fields \mathbf{u} and \mathbf{v} . Show also that $(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla(\frac{1}{2}u^2) - \mathbf{u} \times (\nabla \times \mathbf{u})$.

5 Check, by calculating its curl, that the force field

$$\mathbf{F} = (3x^2 \tan z - y^2 e^{-xy^2} \sin y, (\cos y - 2xy \sin y) e^{-xy^2}, x^3 \sec^2 z)$$

is *conservative*. Find the most general scalar potential for \mathbf{F} and hence, or otherwise, find the work done by the force as it acts on a particle moving from $(0, 0, 0)$ to $(1, \pi/2, \pi/4)$.

6 Verify that the vector field

$$\mathbf{u} = e^x(x \cos y + \cos y - y \sin y) \mathbf{i} + e^x(-x \sin y - \sin y - y \cos y) \mathbf{j}$$

is *irrotational* and express it as the gradient of a scalar field ϕ . Check that \mathbf{u} is also *solenoidal* and show that it can be written as the curl of a vector field $\psi \mathbf{k}$, for some function ψ .

7 (a) The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals to a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that

$$\mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0.$$

7 (b)* The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $|\mathbf{H}|^{-3} |\mathbf{H} \times (\mathbf{H} \cdot \nabla) \mathbf{H}|$.

8 Evaluate the line integral

$$\oint_C -x^2 y \, dx + xy^2 \, dy$$

for C a circle with radius R and centre the origin, traversed anti-clockwise in the xy plane. Use Green's Theorem to express the difference of the results for $R = b$ and $R = a$ in terms of an integral over the region $a^2 \leq x^2 + y^2 \leq b^2$. Evaluate this directly, and check that the results agree.

9 Verify Stokes's Theorem for the hemispherical surface $r = 1, z \geq 0$, and the vector field

$$\mathbf{F}(\mathbf{r}) = (y, -x, z).$$

10 Let $\mathbf{F}(\mathbf{r}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$, and let S be the *open* surface

$$1 - z = x^2 + y^2, \quad 0 \leq z \leq 1.$$

Use the divergence theorem (and cylindrical polar coordinates) to evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$.

Verify your result by calculating the integral directly. [You should find that the vector area element is $d\mathbf{S} = (2\rho \cos \phi, 2\rho \sin \phi, 1) \rho \, d\rho \, d\phi$.]

11 By applying the divergence theorem to the vector field $\mathbf{a} \times \mathbf{A}$, where \mathbf{a} is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field, show that

$$\int_V \nabla \times \mathbf{A} \, dV = - \int_S \mathbf{A} \times d\mathbf{S},$$

where the surface S encloses the volume V .

Verify this result when S is the sphere $|\mathbf{x}| = R$ and $\mathbf{A} = (z, 0, 0)$ in Cartesian coordinates.

12 By applying Stokes's theorem to the vector field $\mathbf{a} \times \mathbf{F}$, where \mathbf{a} is an arbitrary constant vector and $\mathbf{F}(\mathbf{x})$ is a vector field, show that

$$\oint_C d\mathbf{x} \times \mathbf{F} = \int_S (d\mathbf{S} \times \nabla) \times \mathbf{F},$$

where the curve C bounds the open surface S .

Verify this result when C is the unit square in the xy plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$ and $\mathbf{F}(\mathbf{x}) = \mathbf{x}$.

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