Vector Calculus: Example Sheet 2

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1. The vector field \( \mathbf{B}(x) \) is everywhere parallel to the normals of a family of surfaces \( f(x) = \text{constant} \). Show that \( \mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0 \).

   The tangent vector at each point on a curve is parallel to a non-vanishing vector field \( \mathbf{H}(x) \). Show that the curvature of the curve is given by \( \kappa = |\mathbf{H} \times (\mathbf{H} \cdot \nabla)\mathbf{H}|/|\mathbf{H}|^3 \).

2. Let \( \phi(x) \) be a scalar field and \( \mathbf{v}(x) \) a vector field. Show, using suffix notation, that
   \[
   \nabla \cdot (\phi \mathbf{v}) = (\nabla \phi) \cdot \mathbf{v} + \phi (\nabla \cdot \mathbf{v}),
   \nabla \times (\phi \mathbf{v}) = (\nabla \phi) \times \mathbf{v} + \phi (\nabla \times \mathbf{v}).
   \]

   Evaluate the divergence and curl of the following:
   \( r \mathbf{x}, \mathbf{a}(\mathbf{x} \cdot \mathbf{b}), \mathbf{a} \times \mathbf{x}, \mathbf{x}/|\mathbf{x}|^3 \),

   where \( r = |\mathbf{x}| \) and \( \mathbf{a}, \mathbf{b} \) are constant vectors.

3. For vector fields \( \mathbf{u}(x) \) and \( \mathbf{v}(x) \), use suffix notation to show that,
   \[
   \nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u} (\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla) \mathbf{u} - \mathbf{v} (\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla) \mathbf{v},
   \]

   Show also that
   \[
   (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u})
   \]

4. Verify directly that the vector field
   \[
   \mathbf{u}(x) = (e^x (x \cos y + \cos y - y \sin y), e^x (-x \sin y - \sin y - y \cos y), 0)
   \]

   is irrotational and express is as the gradient of a scalar field \( \phi \). Check that \( \mathbf{u} \) is solenoidal and show that it can be written as the curl of the vector field \( \mathbf{v} = (0, 0, \psi) \),

   for some function \( \psi \).

5. Check that the following vector field is irrotational
   \[
   \mathbf{F} = (3x^2 \tan z - y^2 e^{-xy^2} \sin y, (\cos y - 2xy \sin y) e^{-xy^2}, x^3 \sec^2 z)
   \]

   Find the most general scalar potential \( \phi(x) \) such that \( \mathbf{F} = \nabla \phi \).
6*. Suppose \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) is divergence free, i.e. \( \nabla \cdot F = 0 \). Show that \( F = \nabla \times A \) where
\[
A(x) = \int_0^1 F(tx) \times (tx) \, dt.
\]
What goes wrong with this formula if \( F \) is not defined on all of \( \mathbb{R}^3 \)?

7. Let \((u, v, w)\) be a set of orthogonal curvilinear coordinates for \( \mathbb{R}^3 \). Show that
\[
dV = h_u h_v h_w \, du \, dv \, dw.
\]
Confirm that \( dV = \rho \, d\rho \, d\phi \, dz \) and \( dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \) an cylindrical and spherical polars respectively.

8. If \( a \) is constant vector and \( r = |x| \), verify that
\[
\nabla (r^n) = nr^{n-2}x, \quad \nabla (a \cdot x) = a
\]
using (i) Cartesian coordinates and suffix notation, (ii) cylindrical polar coordinates, (iii) spherical polar coordinates, and (iv) a first principles, coordinate-independent approach.

[Note: for parts (ii) and (iii) you will need to be careful with the components of \( a \) with respect to each of the relevant bases.]

9. The vector field \( A(x) \) is, in Cartesian, cylindrical and spherical polar coordinates respectively,
\[
A(x) = -\frac{1}{2} y e_x + \frac{1}{2} x e_y = \frac{1}{2} \rho e_\phi = \frac{1}{2} r \sin \theta e_\phi.
\]
Compute the \( \nabla \times A \) in each different coordinate system and check that your answers agree.

10. Recall that in cylindrical polar coordinates
\[
\nabla = e_\rho \frac{\partial}{\partial \rho} + e_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + e_z \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial e_\rho}{\partial \phi} = e_\phi, \quad \frac{\partial e_\phi}{\partial \phi} = -e_\rho,
\]
while all other derivatives are zero. Derive expressions for the \( \nabla \cdot A \) and \( \nabla \times A \) where \( A \) is an arbitrary vector field given in cylindrical polars by \( A = A_\rho e_\rho + A_\phi e_\phi + A_z e_z \). Also derive an expression for the Laplacian of a scalar function \( \nabla^2 f \) in this coordinate system.
11. By applying the divergence theorem to the vector field \( \mathbf{a} \times \mathbf{A} \), where \( \mathbf{a} \) is an arbitrary constant vector and \( \mathbf{A}(\mathbf{x}) \) is a vector field, show that

\[
\int_V \nabla \times \mathbf{A} \, dV = \int_S \mathbf{d}S \times \mathbf{A}
\]

where \( S = \partial V \). Verify this result when \( V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\} \) and \( \mathbf{A}(\mathbf{x}) = (z, 0, 0) \).

12. Let \( \mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2) \) and let \( S \) be the open surface

\[ 1 - z = x^2 + y^2, \quad 0 \leq z \leq 1. \]

Use the divergence theorem and cylindrical polar coordinates to evaluate \( \int_S \mathbf{F} \cdot \mathbf{d}S \). Verify your result by calculating the area integral directly.

[Hint: you should find that \( \mathbf{d}S = (2\rho \cos \phi, 2\rho \sin \phi, 1) \rho \, d\rho \, d\phi \).]

13. For the electric and magnetic fields \( \mathbf{E}(\mathbf{x}, t) \) and \( \mathbf{B}(\mathbf{x}, t) \) define the quantities

\[
U = \frac{1}{2} \left( \varepsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0} |\mathbf{B}|^2 \right), \quad \mathbf{P} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.
\]

Use Maxwell's equations with \( \mathbf{J} = 0 \) to establish the conservation law \( \partial U/\partial t + \nabla \cdot \mathbf{P} = 0 \). If \( U(\mathbf{x}) \) has the interpretation of the energy density stored in electric and magnetic fields, what is the interpretation of the so-called Poynting vector \( \mathbf{P} \)?