Mathematical Tripos Part IA Lent 2018
Vector Calculus, Example Sheet 2 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Compute the volume of a cone of height h and radius a using (a) cylindrical polars, (b) spherical polars.

2. By using a suitable change of variables, calculate the volume within an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

3. Let (u, v, w) be a set of orthogonal curvilinear coordinates for $\mathbf{R}^3$. Show that

$$\mathrm{d}V = h_u h_v h_w \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w.$$

Confirm that $\mathrm{d}V = \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}z$ and $\mathrm{d}V = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$ an cylindrical and spherical polars respectively.

4. Let $f = f(\mathbf{x})$ be a scalar field and $\mathbf{v} = \mathbf{v}(\mathbf{x})$ a vector field. Show, using suffix notation, that

$$\nabla \cdot (f\mathbf{v}) = (\nabla f) \cdot \mathbf{v} + f(\nabla \cdot \mathbf{v}), \quad \nabla \times (f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f(\nabla \times \mathbf{v}).$$

Evaluate the divergence and curl of the following:

$$r\mathbf{x}, \quad \mathbf{a}(\mathbf{x} \cdot \mathbf{b}), \quad \mathbf{a} \times \mathbf{x}, \quad \mathbf{x}/r^3,$$

where $r = |\mathbf{x}|$ and $\mathbf{a}, \mathbf{b}$ are constant vectors.

5. The vector field $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is, in Cartesian, cylindrical and spherical polar coordinates respectively,

$$\mathbf{A}(\mathbf{x}) = -\tfrac{1}{2} y \, \mathbf{e}_x + \tfrac{1}{2} x \, \mathbf{e}_y = \tfrac{1}{2} \rho \, \mathbf{e}_\phi = \tfrac{1}{2} r \sin\theta \, \mathbf{e}_\phi.$$

Compute the $\nabla \times \mathbf{A}$ in each different coordinate system and check that your answers agree.

6. Recall that in cylindrical polar coordinates

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho,$$

while all other derivatives are zero. Derive expressions for the $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ where $\mathbf{A}$ is an arbitrary vector field given in cylindrical polars by $\mathbf{A} = A_\rho \mathbf{e}_\rho + A_\phi \mathbf{e}_\phi + A_z \mathbf{e}_z$. Also derive an expression for the Laplacian of a scalar function $\nabla^2 f$ in this coordinate system

7. Use suffix notation to show that

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{v},$$

for vector fields $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\mathbf{v} = \mathbf{v}(\mathbf{x})$. Show also that $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left(\tfrac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u})$.

8. Verify directly that the vector field

$$\mathbf{u}(\mathbf{x}) = (e^x (x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0)$$

is *irrotational* and express is as the gradient of a scalar field ϕ. Check that $\mathbf{u}$ is *solenoidal* and show that it can be written as the curl of the vector field $\mathbf{v} = (0, 0, \psi)$, for some function ψ.

9. Consider the line integral

$$I = \oint_C -x^2 y \, \mathrm{d}x + x y^2 \, \mathrm{d}y$$

for C a closed curve traversed anti-clockwise in the (x, y)-plane.

(i) Evaluate I when C is a circle of radius R centred at the origin. Use Green's theorem to relate the results for $R = b$ and $R = a$ to an area integral over an appropriate region, and calculate the area integral directly.

(ii) Now suppose C is the boundary of a square centred at the origin with sides of length ℓ. Show that I is independent of the orientation of the square in the (x, y)-plane.

10. Verify Stokes' theorem for the hemispherical shell $S = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$, and the vector field

$$\mathbf{F}(\mathbf{x}) = (y, -x, z).$$

11. By applying Stokes' theorem to the vector field $\mathbf{a} \times \mathbf{F}$ for $\mathbf{a}$ constant, or otherwise, show that for a vector field $\mathbf{F} = \mathbf{F}(\mathbf{x})$

$$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{F} = \int_S (\mathrm{d}\mathbf{S} \times \nabla) \times \mathbf{F}$$

where $C = \partial S$. Verify this result when C is the unit square in the (x, y)-plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$ and $\mathbf{F}(\mathbf{x}) = \mathbf{x}$.

12. Let $S = \{\mathbf{x} : |\mathbf{x}| = 1\}$. For the vector field $\mathbf{F}(\mathbf{x}) = \mathbf{x}/r^3$, where $r = |\mathbf{x}|$, compute the integral

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Deduce that there *does not* exist a vector potential for $\mathbf{F}$. Compute $\nabla \cdot \mathbf{F}$ and comment on your result.

Additional problems

*These questions should **not** be attempted at the expense of earlier ones.*

13. Compute the volume of the region V defined by the intersection of the three cylinders

$$V = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 \leq 1,\ y^2 + z^2 \leq 1,\ z^2 + x^2 \leq 1\}.$$

14. Let $\mathbf{F}(\mathbf{x}) = \mathbf{x}/r^3$ as in question 12. Consider the vector field

$$\mathbf{A}(\mathbf{x}) = \left(\frac{yz}{(x^2 + y^2)\sqrt{x^2 + y^2 + z^2}}, -\frac{xz}{(x^2 + y^2)\sqrt{x^2 + y^2 + z^2}}, 0 \right).$$

Show that $\nabla \times \mathbf{A} = \mathbf{F}$. Does this contradict the result of question 12? Reconcile these results by applying Stokes' theorem on the surface $S_\epsilon = \{\mathbf{x} : |\mathbf{x}| = 1,\ x^2 + y^2 \geq \epsilon^2\}$ and taking a suitable limit.

Hint: you may find cylindrical or spherical polar coordinates helpful for this question.

15. Suppose $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ is a solenoidal vector field, i.e. $\nabla \cdot \mathbf{F} = 0$. Show that $\mathbf{F} = \nabla \times \mathbf{A}$ where

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \times (t\mathbf{x})\,\mathrm{d}t.$$

This is called a *homotopy formula*. What goes wrong with this formula if $\mathbf{F}$ is not defined on all of $\mathbf{R}^3$?