Mathematical Tripos Part IA **Lent Term 2011**
Vector Calculus **Dr. J.M. Evans**

Example Sheet 3

1 (i) Write down the operator $\boldsymbol{\nabla}$ in Cartesian coordinates and in spherical polars, and calculate the gradient of

$$\psi \,=\, Ez \,=\, Er\cos\theta$$

in both coordinate systems. By considering the relationship between the basis vectors, check that your answers agree. (ii) Calculate, in three ways, the curl of the vector field

$$\mathbf{A} \,=\, \tfrac{1}{2}B(\,-y\,\mathbf{e}_x + x\,\mathbf{e}_y\,) \,=\, \tfrac{1}{2}B\rho\,\mathbf{e}_\phi \,=\, \tfrac{1}{2}Br\sin\theta\,\mathbf{e}_\phi$$

by applying the standard formulas in Cartesian, cylindrical, and spherical polar coordinates.

2 In cylindrical polar coordinates,

$$\boldsymbol{\nabla} \,=\, \mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z} \qquad \text{and} \qquad \frac{\partial\mathbf{e}_\rho}{\partial\phi} = \mathbf{e}_\phi\,, \quad \frac{\partial\mathbf{e}_\phi}{\partial\phi} = -\mathbf{e}_\rho\,,$$

while all other derivatives of the basis vectors are zero. Derive expressions for $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A} = A_\rho\mathbf{e}_\rho + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_z$, and also for $\nabla^2\psi$, where ψ is a scalar field.

3 The vector field $\mathbf{B}(\mathbf{x})$ is defined in cylindrical polar coordinates ρ, ϕ, z by

$$\mathbf{B}(\mathbf{x}) \,=\, \rho^{-1}\mathbf{e}_\phi\,, \qquad\qquad \rho\neq 0\,.$$

Calculate $\boldsymbol{\nabla}\times\mathbf{B}$ using the formula for curl in cylindrical polars. Evaluate $\oint_C \mathbf{B}\cdot d\mathbf{x}$, where C is the circle $z=0$, $\rho=1$ and $0\leqslant\phi\leqslant 2\pi$. Is your answer consistent with Stokes's Theorem?

4 The scalar field $\varphi(r)$ depends only on $r=|\mathbf{x}|$, where $\mathbf{x}$ is the position vector in three dimensions. Use Cartesian coordinates and the chain rule to show that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\,\frac{\mathbf{x}}{r}\,, \qquad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\,\varphi'(r)\,.$$

Find the solution of $\nabla^2\varphi = 1$ which is defined on the region $r\leqslant a$ and which satisfies $\varphi(a)=1$.

5 Using polar coordinates (r,θ), verify that Laplace's equation $\nabla^2\varphi=0$ in two dimensions has solutions $\varphi(r,\theta) = A\,r^\alpha\cos n\theta$, with n an integer, if α is chosen appropriately. Hence solve $\nabla^2\phi = 0$ on the following regions, with the given boundary conditions (where Φ is a constant):

(i) $r\leqslant a\,,\quad \varphi(a,\theta) = \Phi\cos\theta\,;$ (ii) $r\geqslant a\,,\quad \varphi(a,\theta)=\Phi\cos\theta\,,\quad \phi\to 0$ as $r\to\infty\,;$

(iii) $a\leqslant r\leqslant b\,,\quad \dfrac{\partial\varphi}{\partial n}(a,\theta) = 0\,,\quad \varphi(b,\theta) = \Phi\cos 2\theta\,.$

6 Consider a complex-valued function $f \,=\, \varphi(x,y) + i\psi(x,y)$ satisfying $\partial f/\partial\bar z = 0$, where $\partial/\partial\bar z = \partial/\partial x + i\partial/\partial y$. Show that $\nabla^2\varphi = \nabla^2\psi = 0$, and that a curve in the xy plane on which φ is constant intersects orthogonally a curve on which ψ is constant. Find φ and ψ when $f = ze^z$, where $z = x+iy$, and compare with your answers to question **6** on Sheet 2.

7 Let $\rho(\mathbf{x})$ be a function on a volume V and $f(\mathbf{x})$ a function on its boundary $S=\partial V$. Show that a solution $\varphi(\mathbf{x})$ to the following problem is unique:

$$\nabla^2\varphi - \varphi = \rho \;\text{ on }\; V\,, \qquad \frac{\partial\varphi}{\partial n} = f \;\text{ on }\; S\,.$$

8 Show that there is at most one solution $\varphi(\mathbf{x})$ to Laplace's equation in a volume V with the boundary condition given in terms of functions $f(\mathbf{x})$ and $g(\mathbf{x})$ by

$$g\frac{\partial\varphi}{\partial n} + \varphi = f \quad \text{on} \quad \partial V \ ,$$

assuming $g(\mathbf{x}) \geqslant 0$ on ∂V. Find a non-zero solution of Laplace's equation on $|\mathbf{x}| \leqslant 1$ which satisfies the boundary condition above with $f = 0$ and $g = -1$ on $|\mathbf{x}| = 1$.

9 The functions $u(\mathbf{x})$ and $v(\mathbf{x})$ on V satisfy $\nabla^2 u = 0$ on V and $v = 0$ on ∂V. Show that

$$\int_V \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \ \mathrm{d}V \ = \ 0 \ .$$

Now if $w(\mathbf{x})$ is a function on V with $u = w$ on ∂V, show, by considering $v = w - u$, that

$$\int_V |\boldsymbol{\nabla} w|^2 \ \mathrm{d}V \ \geqslant \ \int_V |\boldsymbol{\nabla} u|^2 \ \mathrm{d}V \, .$$

10 Show that Maxwell's equations for $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ imply that the charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{j}(\mathbf{x}, t)$ satisfy the conservation equation $\boldsymbol{\nabla}\cdot\mathbf{j} = -\partial\rho/\partial t$. Show also that if $\mathbf{j}$ is zero then

$$U \ = \ \tfrac{1}{2}\epsilon_0(\,\mathbf{E}^2 + c^2\mathbf{B}^2\,) \quad \text{and} \quad \mathbf{P} \ = \ \mu_0^{-1}\mathbf{E}\times\mathbf{B} \qquad \text{satisfy} \qquad \boldsymbol{\nabla}\cdot\mathbf{P} \ = \ -\partial U/\partial t \ .$$

11 Use Gauss's flux method to find the gravitational field $\mathbf{g}(\mathbf{r})$ due to a spherical shell of matter with density

$$\rho(r) = \begin{cases} 0 & \text{for } 0 < r < a, \\ \rho_0 r/a & \text{for } a < r < b, \\ 0 & \text{for } b < r < \infty. \end{cases}$$

Check your answer by calculating the gravitational potential φ directly, by solving Poisson's equation. You should assume that φ is a function only of r, is not singular at the origin, and that $\varphi(r)$ and $\varphi'(r)$ are continuous at $r = a$ and $r = b$.

12 An object occupies a volume V with boundary a closed surface S. Its *capacity* is defined to be $C = -\int_S \boldsymbol{\nabla}\psi \cdot \mathrm{d}\mathbf{S}$ (using the outward normal) where $\psi(\mathbf{x})$ satisfies Laplace's equation in the region outside V, with boundary conditions $\psi = 1$ on S and $\psi = O(1/|\mathbf{x}|)$ as $|\mathbf{x}| \to \infty$.

(a) Suppose the object carries a static distribution of electric charge, with total charge Q and with zero charge outside V. Assuming the electrostatic potential $\varphi(\mathbf{x})$ is a constant φ_0 on S, show that $C = Q/\epsilon_0\varphi_0$.

(b) Show that the capacity of a sphere of radius R is $4\pi R$.

(c)* Show that the capacity of a cube with edges of length a satisfies $2\pi a < C < 2\sqrt{3}\pi a$. [Hint: Consider a sphere inscribed in the cube, and the cube inscribed in a larger sphere, and use the result of question **9**.]

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