

Example Sheet 3

1 (i) Write down the operator ∇ in Cartesian coordinates and in spherical polars. Calculate the gradient of

$$\psi = Ez = Er \cos \theta$$

in both coordinate systems (E is a constant) and check that your answers agree.

(ii) Apply the standard formulas in Cartesian, cylindrical, and spherical polar coordinates to calculate, in three ways, the curl of the following vector field (with B a constant):

$$\mathbf{A} = \frac{1}{2}B(-y \mathbf{e}_x + x \mathbf{e}_y) = \frac{1}{2}B\rho \mathbf{e}_\phi = \frac{1}{2}Br \sin \theta \mathbf{e}_\phi .$$

2 In cylindrical polar coordinates,

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho,$$

while all other derivatives of the basis vectors are zero. Derive expressions for $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$, where $\mathbf{A} = A_\rho \mathbf{e}_\rho + A_\phi \mathbf{e}_\phi + A_z \mathbf{e}_z$, and also for $\nabla^2 \psi$, where ψ is a scalar field.

3 The vector field $\mathbf{B}(\mathbf{x})$ is defined in cylindrical polar coordinates ρ, ϕ, z by

$$\mathbf{B}(\mathbf{x}) = \rho^{-1} \mathbf{e}_\phi, \quad \rho \neq 0 .$$

Calculate $\nabla \times \mathbf{B}$ using the formula for curl in cylindrical polars. Evaluate $\oint_C \mathbf{B} \cdot d\mathbf{x}$, where C is the circle $z = 0, \rho = 1$ and $0 \leq \phi \leq 2\pi$. Is your answer consistent with Stokes's Theorem?

4 The scalar field $\varphi(r)$ depends only on $r = |\mathbf{x}|$, where \mathbf{x} is the position vector in three dimensions. Use Cartesian coordinates, index notation, and the chain rule to show that

$$\nabla \varphi = \varphi'(r) \frac{\mathbf{x}}{r}, \quad \nabla^2 \varphi = \varphi''(r) + \frac{2}{r} \varphi'(r).$$

Find the solution of $\nabla^2 \varphi = 1$ which is defined on the region $r \leq a$ and which satisfies $\varphi(a) = 1$.

5 (a) Using Cartesian coordinates x, y , find all solutions of Laplace's equation in two dimensions of the form $\varphi(x, y) = f(x)e^{\alpha y}$ with α a constant. Hence find a solution on the region $0 \leq x \leq a$ and $y \geq 0$ with boundary conditions:

$$\varphi(0, y) = \varphi(a, y) = 0, \quad \varphi(x, 0) = \lambda \sin(\pi x/a), \quad \varphi \rightarrow 0 \text{ as } y \rightarrow \infty \quad (\lambda \text{ a const}).$$

(b) Using the formula for ∇^2 in polar coordinates r, θ , verify that Laplace's equation in the plane has solutions $\varphi(r, \theta) = Ar^\alpha \cos \beta \theta$, if α and β are related appropriately. Hence find solutions on the following regions, with the given boundary conditions (λ a const):

$$(i) \quad r \leq a, \quad \varphi(a, \theta) = \lambda \cos \theta; \quad (ii) \quad r \geq a, \quad \varphi(a, \theta) = \lambda \cos \theta, \quad \varphi \rightarrow 0 \text{ as } r \rightarrow \infty;$$

$$(iii) \quad a \leq r \leq b, \quad \frac{\partial \varphi}{\partial n}(a, \theta) = 0, \quad \varphi(b, \theta) = \lambda \cos 2\theta .$$

6 Consider a complex-valued function $f = \varphi(x, y) + i\psi(x, y)$ satisfying $\partial f / \partial \bar{z} = 0$, where $\partial / \partial \bar{z} = \partial / \partial x + i\partial / \partial y$. Show that $\nabla^2 \varphi = \nabla^2 \psi = 0$. Show also that a curve on which φ is constant is orthogonal to a curve on which ψ is constant at a point where they intersect. Find φ and ψ when $f = ze^z$, where $z = x + iy$, and compare with question 6 on Sheet 2.

7 Use Gauss's flux method to find the gravitational field $\mathbf{g}(\mathbf{r})$ due to a spherical shell of matter with density

$$\rho(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq a, \\ \rho_0 r/a & \text{for } a < r < b, \\ 0 & \text{for } r \geq b. \end{cases}$$

Now find the gravitational potential $\varphi(r)$ *directly* from Poisson's equation by writing down the general, spherically symmetric solution to Laplace's equation in each of the intervals $0 < r < a$, $a < r < b$ and $r > b$, and adding a particular integral where necessary. Assume that φ is not singular at the origin, and that φ and φ' are continuous at $r = a$ and $r = b$. Check that this solution gives the same result for the gravitational field.

8 From an integral theorem, derive (one of *Green's Identities*):

$$\int_V (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) dV = \int_{\partial V} \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) dS .$$

9 Let $\rho(\mathbf{x})$ be a function on a volume V and $f(\mathbf{x})$ a function on its boundary $S = \partial V$. Show that a solution $\varphi(\mathbf{x})$ to the following problem is unique:

$$\nabla^2 \varphi - \varphi = \rho \quad \text{on } V, \quad \frac{\partial \varphi}{\partial n} = f \quad \text{on } S .$$

10 The functions $u(\mathbf{x})$ and $v(\mathbf{x})$ on V satisfy $\nabla^2 u = 0$ on V and $v = 0$ on ∂V . Show that

$$\int_V \nabla u \cdot \nabla v dV = 0 .$$

Now if $w(\mathbf{x})$ is a function on V with $u = w$ on ∂V , show, by considering $v = w - u$, that

$$\int_V |\nabla w|^2 dV \geq \int_V |\nabla u|^2 dV .$$

11 Show that there is at most one solution $\varphi(\mathbf{x})$ to Laplace's equation in a volume V with the boundary condition given in terms of functions $f(\mathbf{x})$ and $g(\mathbf{x})$ by

$$g \frac{\partial \varphi}{\partial n} + \varphi = f \quad \text{on } \partial V ,$$

assuming $g(\mathbf{x}) \geq 0$ on ∂V . Find a non-zero solution of Laplace's equation on $|\mathbf{x}| \leq 1$ which satisfies the boundary condition above with $f = 0$ and $g = -1$ on $|\mathbf{x}| = 1$.

12 Maxwell's equations for electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho / \epsilon_0 , & \nabla \times \mathbf{E} &= -\partial \mathbf{B} / \partial t , \\ \nabla \cdot \mathbf{B} &= 0 , & \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \partial \mathbf{E} / \partial t , \end{aligned}$$

where $\rho(\mathbf{x}, t)$ and $\mathbf{j}(\mathbf{x}, t)$ are the charge density and current, and ϵ_0 and μ_0 are constants. Show that these imply the conservation equation $\nabla \cdot \mathbf{j} = -\partial \rho / \partial t$. Show also that if \mathbf{j} is zero then

$$U = \frac{1}{2} (\epsilon_0 \mathbf{E}^2 + \mu_0^{-1} \mathbf{B}^2) \quad \text{and} \quad \mathbf{P} = \mu_0^{-1} \mathbf{E} \times \mathbf{B} \quad \text{satisfy} \quad \nabla \cdot \mathbf{P} = -\partial U / \partial t .$$

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