Vector Calculus: Example Sheet 3

Part IA, Lent Term 2025

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Integral Theorems

1. Verify Stokes' theorem for the hemispherical shell $S = \{x^2 + y^2 + z^2 = 1, z \geqslant 0\}$ and the vector field $\mathbf{F}(\mathbf{x}) = (y, -x, z)$.

2. Verify Stokes' theorem for the open surface defined in cylindrical polar coordinates by $\rho + z = a$, $z \geqslant 0$, where a is a positive constant, and the vector field $\mathbf{F}(\mathbf{x}) = (y, -x, xyz)$.

3. Calculate $\nabla \times \mathbf{B}$ for the vector field given in cylindrical polars by $\mathbf{B}(\mathbf{x}) = \rho^{-1}\mathbf{e}_\phi$. Also find $\oint_C \mathbf{B} . \mathrm{d}\mathbf{x}$ where C is the circle $\rho = 1$, $z = 0$. Does Stokes' Theorem apply?

4. Let $S = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$. For the vector field $\mathbf{F}(\mathbf{x}) = \mathbf{x}/r^3$, where $r = |\mathbf{x}|$, compute the integral $\int_S \mathbf{F} . \mathrm{d}\mathbf{S}$. Deduce that there *does not* exist a vector potential for $\mathbf{F}$, i.e., there is no vector field $\mathbf{A}(\mathbf{x})$ such that $\mathbf{F} = \nabla \times \mathbf{A}$. Compute $\nabla . \mathbf{F}$ and comment on your result.

5. Let ϕ and ψ be scalar functions defined in a volume V and on its surface. Using an integral theorem, establish *Green's second identity*

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, \mathrm{d}V = \int_{\partial V} \left(\psi \frac{\partial \phi}{\partial \mathbf{n}} - \phi \frac{\partial \psi}{\partial \mathbf{n}} \right) \mathrm{d}S.$$

6. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$ and let S be the *open* surface $1 - z = x^2 + y^2$, $0 \leqslant z \leqslant 1$. Sketch S, then use the divergence theorem and cylindrical polar coordinates to evaluate $\int_S \mathbf{F} . \mathrm{d}\mathbf{S}$. Verify your result by calculating the area integral directly.

7. By applying the divergence theorem to $\mathbf{k} \times \mathbf{A}$, where $\mathbf{k}$ is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field defined in a volume V and on its surface, show that

$$\int_V \nabla \times \mathbf{A}\, \mathrm{d}V = \int_{\partial V} \mathrm{d}\mathbf{S} \times \mathbf{A}.$$

Verify this result when $V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and $\mathbf{A}(\mathbf{x}) = (z, 0, 0)$. [*You should find that both integrals equal* $(0, abc, 0)$.]

8. By applying Stokes' theorem to the vector field $\mathbf{a} \times \mathbf{F}$ for any constant vector $\mathbf{a}$, or otherwise, show that for a vector field $\mathbf{F}(\mathbf{x})$,

$$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{F} = \int_S (\mathrm{d}\mathbf{S} \times \nabla) \times \mathbf{F}$$

where S is any suitable surface and $C = \partial S$. Verify this result when $\mathbf{F}(\mathbf{x}) = \mathbf{x}$ and C is the unit square in the (x, y)-plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$.

* **9.** Let f be a scalar field and $\mathbf{F}$ a vector field defined on a volume V. By applying the divergence theorem to suitable vector fields, prove that:

(i) If f is constant on ∂V then $\int_V \nabla f\, \mathrm{d}V = \mathbf{0}$.

(ii) If $\nabla . \mathbf{F} = 0$ in V and $\mathbf{F} . \mathbf{n} = 0$ on ∂V then $\int_V \mathbf{F}\, \mathrm{d}V = \mathbf{0}$.

Conservation Laws

10. For electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, define the quantities $U = \frac{1}{2}(\epsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2)$ and $\mathbf{P} = \mu_0^{-1}\mathbf{E}\times\mathbf{B}$. Use Maxwell's equations with $\mathbf{J} = \mathbf{0}$ to establish the conservation law $\partial U/\partial t + \nabla\cdot\mathbf{P} = 0$.

If U has the interpretation of the energy density stored in the electromagnetic fields, what is the interpretation of the *Poynting vector* $\mathbf{P}$?

Laplace's and Poisson's Equations

11. The scalar field $\varphi = \varphi(r)$ only depends on $r = |\mathbf{x}|$ in $\mathbb{R}^3$. Use Cartesian coordinates and suffix notation to show that

$$\nabla\varphi = \varphi'(r)\frac{\mathbf{x}}{r}, \quad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\varphi'(r).$$

Verify this result using your expression for the Laplacian in spherical polar coordinates.

Solve the equation $\nabla^2\varphi = 1$ in $r < a$, subject to $\varphi = 1$ on $r = a$.

What are the equivalent results in $\mathbb{R}^2$?

12. (i) Using Cartesian coordinates (x,y), find all solutions of Laplace's equation $\nabla^2\phi = 0$ in two dimensions of the form $\phi(x,y) = f(x)e^{\alpha y}$, with α constant. Hence find a solution on the region $0 < x < a$, $y > 0$ with boundary conditions

$$\phi(0,y) = \phi(a,y) = 0, \quad \phi(x,0) = \sin(\pi x/a), \quad \phi(x,y) \to 0 \text{ as } y \to \infty.$$

(ii) Using the formula for the Laplacian in plane polar coordinates (r,θ), verify that Laplace's equation in the plane has solutions of the form $\phi(r,\theta) = Ar^\alpha\cos\beta\theta$ if α and β are related appropriately. Hence find solutions on the following regions:

(a) $r < a$, subject to $\phi(a,\theta) = \cos\theta$;

(b) $r > a$, subject to $\phi(a,\theta) = \cos\theta$, $\phi(r,\theta) \to 0$ as $r \to \infty$;

(c) $a < r < b$, subject to $\partial\phi/\partial\mathbf{n} = 0$ on $r = a$, $\phi(b,\theta) = \cos 2\theta$.

13. Use Gauss' flux method to find the electric field due to a spherically symmetric charge density

$$\rho(r) = \begin{cases} 0 & 0 \leqslant r \leqslant a, \\ \rho_0 r/a & a < r < b, \\ 0 & r \geqslant b. \end{cases}$$

Now find the electric potential $\phi(r)$ *directly* from Poisson's equation by writing down the general spherically symmetric solution to Laplace's equation in each of the three intervals, and adding a particular integral where necessary. You should assume that ϕ and ϕ' are continuous at $r = a$ and $r = b$. Check that this solution gives rise to the same electric field using $\mathbf{E} = -\nabla\phi$.

14. Let u be harmonic in a domain $\mathscr{D}$ and v be a smooth function satisfying $v = 0$ on $\partial\mathscr{D}$. Show that

$$\int_{\mathscr{D}} \nabla u\cdot\nabla v \,\mathrm{d}V = 0.$$

Now if w is any smooth function in $\mathscr{D}$ with $w = u$ on $\partial\mathscr{D}$, show, by considering $v = w - u$, that

$$\int_{\mathscr{D}} |\nabla w|^2 \,\mathrm{d}V \geqslant \int_{\mathscr{D}} |\nabla u|^2 \,\mathrm{d}V;$$

that is to say, the solution of Laplace's equation minimises this integral.

* **15.** Show that at any point $\mathbf{a}$, a given harmonic function ϕ is equal to the average of its values in the ball $B_r(\mathbf{a}) = \{\mathbf{x} : |\mathbf{x}-\mathbf{a}| < r\}$, for any $r > 0$. By using this result for large r and considering $\nabla\phi$, or otherwise, prove that if ϕ is bounded and harmonic on $\mathbb{R}^3$ then it is constant.