1. Let \( \mathbf{F}(x) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2) \) and let \( S \) be the open surface
\[
1 - z = x^2 + y^2, \quad 0 \leq z \leq 1.
\]
Use the divergence theorem and cylindrical polar coordinates to evaluate \( \int_S \mathbf{F} \cdot d\mathbf{S} \). Verify your result by calculating the area integral directly. *Hint: you should find that* \( d\mathbf{S} = (2\rho \cos \phi, 2\rho \sin \phi, 1) \rho \, d\rho \, d\phi \).

2. By applying the divergence theorem to the vector field \( \mathbf{a} \times \mathbf{A} \), where \( \mathbf{a} \) is an arbitrary constant vector and \( \mathbf{A} = \mathbf{A}(x) \) is a vector field, show that
\[
\int_S \nabla \times \mathbf{A} \, dV = \int_S d\mathbf{S} \times \mathbf{A}
\]
where \( S = \partial V \). Verify this result when \( V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\} \) and \( \mathbf{A}(x) = (z, 0, 0) \).

3. The scalar field \( \varphi = \varphi(r) \) only depends on \( r = |x| \). Use Cartesian coordinates and suffix notation to show
\[
\nabla \varphi = \varphi'(r) \frac{x}{r}, \quad \nabla^2 \varphi = \varphi''(r) + \frac{2}{r} \varphi'(r).
\]
Verify this result using your expression for the Laplacian in spherical polar coordinates. Solve the equation
\[
\begin{aligned}
\nabla^2 \varphi &= 1, \quad r < a \\
\varphi &= 1, \quad r = a.
\end{aligned}
\]

4. (a) Using Cartesian coordinates \((x, y)\), find all solutions of Laplace’s equation \( \nabla^2 \varphi = 0 \) in two dimensions of the form \( \varphi(x, y) = f(x)e^{\alpha y} \), with \( \alpha \) constant. Hence find a solution on the region \( 0 < x < a \) and \( y > 0 \) with boundary conditions:
\[
\varphi(0, y) = \varphi(a, y) = 0, \quad \varphi(x, 0) = \lambda \sin(\pi x/a), \quad \varphi(x, y) \to 0 \text{ as } y \to \infty.
\]
(b) Using the formula for the Laplacian in plane polar coordinates \((r, \theta)\), verify that Laplace’s equation in the plane has solutions of the form \( \varphi(r, \theta) = Ar^\alpha \cos \beta \theta \), if \( \alpha \) and \( \beta \) are related appropriately. Hence find solutions on the following regions, with the given boundary conditions (\( \lambda \) a constant):

(i) \( r < a \), \quad \varphi(a, \theta) = \lambda \cos \theta, \\
(ii) \( r > a \), \quad \varphi(a, \theta) = \lambda \cos \theta, \quad \varphi(r, \theta) \to 0 \text{ as } r \to \infty, \\
(iii) \( a < r < b \), \quad \frac{\partial \varphi}{\partial r}(a, \theta) = 0, \quad \varphi(b, \theta) = \lambda \cos 2\theta.

5. Consider a complex valued function \( f = \varphi(x, y) + i\psi(x, y) \) satisfying \( \partial f/\partial \bar{z} = 0 \), where \( \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \). Show that \( \nabla^2 \varphi = \nabla^2 \psi = 0 \). Show also that a curve on which \( \psi \) is constant is orthogonal to a curve on which \( \psi \) is constant, at a point where they intersect. Find \( \varphi \) and \( \psi \) when \( f = ze^z, \quad z = x + iy \), and compare with question 8 on sheet 2.

6. Use Gauss’ flux method to find the electric field \( \mathbf{E} = \mathbf{E}(x) \) due to a spherically symmetric charge density
\[
\rho(r) = \begin{cases} 
0, & 0 \leq r \leq a \\
\rho_0 r/a, & a < r < b, \\
0, & r \geq b.
\end{cases}
\]

Now find the electric potential \( \phi = \phi(r) \) directly from Poisson’s equation by writing down the general, spherically symmetric solution to Laplace’s equation in each of the intervals \( 0 < r < a, \ a < r < b \) and \( r > b \), and adding a particular integral where necessary. You should assume that \( \phi \) and \( \phi' \) are continuous at \( r = a \) and \( r = b \). Check this solution gives rise to the same electric field using \( \mathbf{E} = -\nabla \phi \).
7. For the electric and magnetic fields $\mathbf{E} = \mathbf{E}(x, t)$ and $\mathbf{B} = \mathbf{B}(x, t)$ define the quantities

$$U = \frac{1}{2} \left( \varepsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0} |\mathbf{B}|^2 \right), \quad \mathbf{P} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$ 

Use Maxwell's equations with $\mathbf{J} = 0$ to establish the conservation law $\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{P} = 0$.

8. Let $\varphi$ and $\psi$ be scalar functions. Using an integral theorem, establish *Green's second identity*

$$\int_V (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) \, dV = \int_{\partial V} (\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}) \, dS.$$ 

9. Show that the solution to the following boundary value problem is unique

$$\begin{cases} -\nabla^2 \varphi + \varphi = \rho, & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial \Omega, \\ \frac{\partial \varphi}{\partial n} = f, & \text{on } \partial \Omega. \end{cases}$$

10. Show that the solution to the following boundary value problem is unique

$$\begin{cases} \nabla^2 \varphi = 0, & \text{in } \Omega, \\ g(\partial \varphi/\partial n) + \varphi = f, & \text{on } \partial \Omega, \end{cases}$$

assuming that $g \geq 0$ on $\partial \Omega$. Find a non-zero solution to Laplace’s equation on $|x| \leq 1$ which satisfies the boundary conditions above with $f = 0$ and $g = -1$ on $|x| = 1$.

11. Let $u$ be harmonic on $\Omega$ and $v$ a smooth function that satisfies $v = 0$ on $\partial \Omega$. Show that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dV = 0.$$ 

Now if $w$ is any function on $\Omega$ with $w = u$ on $\partial \Omega$, show, by considering $v = w - u$, that

$$\int_{\Omega} |\nabla w|^2 \, dV \geq \int_{\Omega} |\nabla u|^2 \, dV.$$ 

**Additional problems**

*These questions should not be attempted at the expense of earlier ones.*

12. For $\epsilon > 0$ define $\Phi_\epsilon(x) = (|x| + \epsilon)^{-1}$. Show that

$$\nabla^2 \Phi_\epsilon(x) = \frac{-2\epsilon}{|x| (|x| + \epsilon)^3}.$$ 

If $\varphi$ is a scalar function that decays rapidly as $|x| \to \infty$ and $a \in \mathbb{R}^3$ is fixed, compute the limit

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} \varphi(x) \nabla^2 \Phi_\epsilon(x - a) \, dV.$$ 

Deduce that $\nabla^2 \left( -\frac{1}{4\pi} \frac{1}{|x - a|} \right) = \delta(x - a)$.

13. Show that a harmonic function $\varphi$ at the point $a$ is equal to the average of its values on the interior of the ball $B_r(a) = \{x : |x - a| < r\}$, for any $r > 0$. By considering $\nabla \varphi$ and the previous result for large $r$, or otherwise, prove that if $\varphi$ is bounded and harmonic on $\mathbb{R}^3$ then it is constant.

14. (Harder) For a volume $V$ with smooth boundary $S$, establish the identity $\text{vol}(V) = \frac{1}{3} \int_S \mathbf{v} \cdot dS$. Suppose now that $V = V(t)$, and the velocity of a point $x$ in $V$ is $\mathbf{v}(x)$. Show that

$$\frac{d}{dt} \text{vol}(V(t)) = \int_{V(t)} \frac{\partial \rho}{\partial t} \, dV + \int_{S(t)} \rho \mathbf{v} \cdot dS.$$ 

Using this result, or otherwise, obtain *Reynold’s Transport Theorem* for a scalar function $\rho = \rho(x, t)$:

$$\frac{d}{dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \frac{\partial \rho}{\partial t} \, dV + \int_{S(t)} \rho \mathbf{v} \cdot dS.$$ 

Interpret this result.