Vector Calculus: Example Sheet 4

Part IA, Lent Term 2025

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Uniqueness of Solutions to Laplace's and Poisson's Equations

1. Show that the solution to the following boundary value problem in a domain $\mathscr{D}$ is unique:

$$-\nabla^2\phi + \phi = \rho \text{ in } \mathscr{D}, \quad \partial\phi/\partial\mathbf{n} = f \text{ on } \partial\mathscr{D}$$

where ρ and f are given functions on $\mathscr{D}$ and $\partial\mathscr{D}$ respectively.

2. Given a volume V with surface S, show that the solution to the following boundary value problem is unique:

$$\nabla^2\psi = 0 \text{ in } V, \quad g\frac{\partial\psi}{\partial\mathbf{n}} + \psi = f \text{ on } S$$

where $f(\mathbf{x})$ and $g(\mathbf{x})$ are specified functions defined on S obeying $g \geqslant 0$.

Show that uniqueness may not hold if the condition $g \geqslant 0$ is not enforced, by finding a nonzero solution to Laplace's equation on $|\mathbf{x}| \leqslant 1$ which satisfies $\psi = \partial\psi/\partial\mathbf{n}$ on $|\mathbf{x}| = 1$.

Cartesian Tensors

3. The current $\mathbf{J}$ within an anisotropic conductive material due to an electric field $\mathbf{E}$ is given by $J_i = \sigma_{ij}E_j$, where σ_{ij} is the conductivity tensor. In a given Cartesian coordinate system,

$$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current is largest, for an electric field of fixed magnitude.

4. Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector and let $P_{ij} = \delta_{ij} - n_i n_j$. Explain why P_{ij} is a tensor. Find its eigenvalues and eigenvectors.

5. Using the transformation law for a second-rank tensor T_{ij}, show that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are scalars, i.e., remain the same in all Cartesian coordinate systems. If T_{ij} is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are the roots of the cubic equation

$$\lambda^3 - \alpha\lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

6. The velocity field $\mathbf{u}(\mathbf{x}, t)$ of an inviscid compressible gas obeys

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \quad \text{and} \quad \rho\Big(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) = -\nabla p,$$

where $\rho(\mathbf{x}, t)$ is the density and $p(\mathbf{x}, t)$ is the pressure. Show that

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho u_i u_i) + \frac{\partial}{\partial x_i}(\tfrac{1}{2}\rho u_j u_j u_i + p u_i) = p\nabla\cdot\mathbf{u} \quad \text{and} \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial t_{ij}}{\partial x_j} = 0$$

for a suitable symmetric tensor t_{ij}, to be determined.

* What interpretation might be given to t_{ij}?

7. If $\mathbf{u}(\mathbf{x})$ is a vector field, show that $\partial u_i/\partial x_j$ transforms as a second rank tensor field. If $\sigma_{ij}(\mathbf{x})$ is a tensor field, show that $\partial \sigma_{ij}/\partial x_j$ transforms as a vector field.

Symmetries of Tensors

8. (i) By first decomposing into symmetric and anti-symmetric parts, show that an arbitrary second rank tensor T_{ij} can be written in the form

$$T_{ij} = \alpha\delta_{ij} + \epsilon_{ijk}\omega_k + B_{ij}$$

where α is a *scalar*, $\boldsymbol{\omega}$ a *vector* and B_{ij} a symmetric traceless second rank *tensor*.

(ii) Let $\mathbf{u}(\mathbf{x})$ be a vector field and let $T_{ij} = \partial u_i/\partial x_j$. Show that $\alpha = \frac{1}{3}\nabla\cdot\mathbf{u}$ and $\boldsymbol{\omega} = -\frac{1}{2}\nabla\times\mathbf{u}$. Find B_{ij} in the case $\mathbf{u}(\mathbf{x}) = (xy^2, yz^2, zx^2)$: verify that $(0,0,1)$ is a principal axis at the point $\mathbf{x} = (2,3,0)$ and find the others.

9. (i) A tensor of rank 3 satisfies $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{ikj}$. Show that $T_{ijk} = 0$.

* (ii) A tensor of rank 4 satisfies $T_{jikl} = -T_{ijkl} = T_{ijlk}$ and $T_{ijij} = 0$. Show that $T_{ijkl} = \epsilon_{ijp}\epsilon_{klq}S_{pq}$ where $S_{pq} = -T_{rqrp}$. [*Hint: Try to generalise a technique pertaining to antisymmetric second rank tensors.*]

10. The array T_{ijk} is such that $T_{ijk}S_{jk}$ is a vector for every symmetric second rank tensor S_{jk}. Show by counterexample that T_{ijk} need not be a tensor; but show also that $T_{ijk} + T_{ikj}$ must be.

11. (i) In a given basis $\mathscr{B}$ comprising three orthogonal axes, the physical shape and properties of a particular body are unchanged by rotations through an angle π about each of the axes. Show that any second rank tensor calculated for the body will be diagonal in $\mathscr{B}$, although the diagonal elements need not be equal.

(ii) A cuboid of uniform density and mass M has sides of length $2a$, $2b$ and $2c$. Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

(iii) In the case of a cube, with $a = b = c$, which is rotating with angular velocity $\boldsymbol{\omega}$ about an axis which passes through a pair of opposite vertices, find its angular momentum.

12. Evaluate the following integrals in $\mathbb{R}^3$, where $\gamma > 0$: $\displaystyle\int_{\mathbb{R}^3} r^{-3}e^{-\gamma r^2}x_i x_j\,\mathrm{d}V$; $\displaystyle\int_{r<1} r^{-5}e^{-\gamma r^2}x_i x_j x_k\,\mathrm{d}V$.

13. In linear elasticity, the symmetric second rank stress tensor σ_{ij} (representing the forces on an elastic medium) depends on the symmetric second rank strain tensor e_{kl} (representing the corresponding deformation) according to $\sigma_{ij} = c_{ijkl}e_{kl}$. Assuming that c_{ijkl} is symmetric in k and l, explain why it must be a fourth rank tensor. For an isotropic medium, use the most general possible form for c_{ijkl} (which you may quote) to show that

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

where λ and μ are scalars. Invert this equation to express e_{ij} in terms of σ_{ij}, assuming that $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of σ_{ij} and e_{ij} coincide.

* **14.** Let $(x_0, x_1, x_2, x_3) \equiv (t, x_1, x_2, x_3)$ be coordinates for Minkowski space $\mathbb{R}^{1+3}$. Show that the two Maxwell equations $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, $\nabla\cdot\mathbf{B} = 0$, are equivalent to

$$\frac{\partial F_{ab}}{\partial x_c} + \frac{\partial F_{bc}}{\partial x_a} + \frac{\partial F_{ca}}{\partial x_b} = 0 \quad \text{where} \quad (F_{ab}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

* **15.** Let $T_{ij\ldots k}$ be an mth rank tensor on $\mathbb{R}^n$ (i.e., each of the m indices can take on n different values). How many independent components does $T_{ij\ldots k}$ have if it is (i) totally antisymmetric; or (ii) totally symmetric?