Vector Calculus: Example Sheet 4

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1. The current $J$ due to an electric field $E$ is given by $J_i = \sigma_{ij}E_j$, where $\sigma_{ij}$ is the conductivity tensor. In a given Cartesian coordinate system

$$ \sigma_{ij} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} $$

Show that there is a direction along which no current flows, and find the direction(s) along which the current is largest, for an electric field of fixed magnitude.

2. Given the vectors $u = (1, 0, 1)$, $v = (0, 1, -1)$ and $w = (1, 1, 0)$, find all components of the second-rank and third-rank tensors defined by

$$ T_{ij} = u_i v_j + v_i w_j; \quad S_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k + w_i u_j v_k - u_i v_j w_k. $$

3. Use the transformation law for a second-rank tensor $T_{ij}$, show that the quantities

$$ \alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \gamma = T_{ij}T_{jk}T_{ki} $$

are scalars, i.e. remain the same in all Cartesian coordinate systems. If $T_{ij}$ is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are the roots of the cubic equation

$$ \lambda^3 - \alpha \lambda^2 + \frac{1}{2} (\alpha^2 - \beta) \lambda - \frac{1}{6} (\alpha^3 - 3\alpha \beta + 2\gamma) = 0. $$

4. If $u_i(x)$ is a vector field, show that $\partial u_i / \partial x_j$ transforms as a second rank tensor field. If $\sigma_{ij}(x)$ is a tensor field, show that $\partial \sigma_{ij} / \partial x_j$ transforms as a vector field.

5. The electric field $E(x, t)$ and magnetic field $B(x, t)$ satisfy Maxwell’s equations with zero charge and current. Show that the Poynting vector $P = \frac{1}{\mu_0} E \times B$ satisfies the conservation law

$$ \frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{\mu_0} \left( \frac{1}{2} \delta_{ij} (E^2 + c^2 B^2) - (E_i E_j + c^2 B_i B_j) \right) $$

where $c^2 = 1/\mu_0 \varepsilon_0$. If the component $P_i$ is the momentum density in the $x^i$ direction stored in the electric and magnetic fields, what is the interpretation of $T_{ij}$?
6. The velocity field \( u(x, t) \) of an inviscid compressible gas obeys

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad \text{and} \quad \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = -\nabla p
\]

where \( \rho(x, t) \) is the density and \( p(x, t) \) is the pressure. Show that

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u_j u_j \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho u_j u_j u_i + p u_i \right) = p \nabla \cdot u \quad \text{and} \quad \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial t_{ij}}{\partial x_j} = 0
\]

for a suitable symmetric tensor \( t_{ij} \), to be determined.

7. By first decomposing into symmetric and anti-symmetric parts, show that an arbitrary second rank tensor \( T_{ij} \) can be written in the form

\[
T_{ij} = \alpha \delta_{ij} + \epsilon_{ijk} \omega_k + B_{ij}
\]

where \( \alpha \) is a scalar, \( \omega_k \) a vector and \( B_{ij} \) a symmetric second rank tensor satisfying \( B_{ii} = 0 \).

(i) A tensor of rank 3 satisfies \( T_{ijk} = T_{jik} \) and \( T_{ijk} = -T_{ikj} \). Show that \( T_{ijk} = 0 \).

(ii) A tensor of rank 4 satisfies \( T_{jikl} = T_{ijkl} = T_{ijlk} \) and \( T_{ijij} = 0 \). Show that

\[
T_{ijkl} = \epsilon_{ijp} \epsilon_{klq} S_{pq} \quad \text{where} \quad S_{pq} = -T_{rqrp}.
\]

8. A cuboid of uniform density and mass \( M \) has sides of length \( 2a, 2b \) and \( 2c \). Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

A cube of sides of length \( 2a \) has uniform charge density, mass \( M \), and is rotating with angular velocity \( \omega \) about an axis which passes through its center and through a pair of opposite vertices. What is its angular momentum?

9. Evaluate the following integrals, where \( \gamma > 0 \) and \( r^2 = x_p x_p \):

(i) \[
\int_{\mathbb{R}^3} r^{-3} e^{-\gamma r^2} x_i x_j \, dV,
\]

(ii) \[
\int_{\mathbb{R}^3} r^{-5} e^{-\gamma r^2} x_i x_j x_k \, dV.
\]

10. A tensor has components \( T_{ij} \) with respect to a given Cartesian coordinate system \( \{x_i\} \). If the tensor is invariant under arbitrary rotations about the \( x_3 \)-axis, show that it must have the form

\[
T_{ij} = \begin{pmatrix}
\alpha & \omega & 0 \\
-\omega & \alpha & 0 \\
0 & 0 & \beta
\end{pmatrix}
\]
11. In the theory of linear elasticity, the symmetric stress tensor $\sigma_{ij}$ depends on the symmetric strain tensor $e_{kl}$ through the equation $\sigma_{ij} = c_{ijkl}e_{kl}$. Explain why $c_{ijkl}$ must be a fourth rank tensor with $c_{ijkl} = c_{jikl}$ and why it may be chosen such that $c_{ijkl} = c_{ijlk}$. For an isotropic medium, use the most general possible form for $c_{ijkl}$ (which you may quote) to show that

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2 \mu e_{ij},$$

where $\lambda$ and $\mu$ are scalars. Invert this equation to express $e_{ij}$ in terms of $\sigma_{ij}$, assuming $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of $\sigma_{ij}$ and $e_{ij}$ coincide.

The elastic energy density resulting from a deformation of the medium is $E = \frac{1}{2} e_{ij} \sigma_{ij}$. Show that $E$ is strictly positive for any non-zero strain $e_{ij}$ provided $\mu > 0$ and $\lambda > -\frac{2}{3} \mu$.

12*. The Levi-Civita tensor of rank $n$ is defined by

$$\epsilon_{i_1 i_2 \cdots i_n} = \begin{cases} +1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an even permutation of } (1, 2, \ldots, n), \\
-1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an odd permutation of } (1, 2, \ldots, n), \\
0, & \text{otherwise}. \end{cases}$$

This is the $n$-dimensional analogue of $\epsilon_{ijk}$. Show that $\epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} = n!$.

How might you get a computer to compute $\epsilon_{i_1 \cdots i_n}$ for a given permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$? For instance, computing the determinant of the relevant permutation matrix is one option, but is quite computationally complex. Can you do better?

13*. Let $T_{i_1 \cdots k}$ be a tensor of $m$ on $\mathbb{R}^n$, meaning that $i, j, \ldots, k = 1, \ldots, n$. How many independent components does $T_{i_1 \cdots k}$ have if it is (a) totally antisymmetric; or (b) totally symmetric?