Vector Calculus: Example Sheet 4

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1. The current $\mathbf{J}$ due to an electric field $\mathbf{E}$ is given by $J_i = \sigma_{ij}E_j$, where $\sigma_{ij}$ is the conductivity tensor. In a given Cartesian coordinate system

$$\sigma_{ij} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current is largest, for an electric field of fixed magnitude.

2. Given the vectors $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (0, 1, -1)$ and $\mathbf{w} = (1, 1, 0)$, find all components of the second-rank and third-rank tensors defined by

$$T_{ij} = u_i v_j + v_i w_j; \quad S_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k - w_i v_j u_k + w_i u_j v_k - u_i w_j v_k.$$

3. Use the transformation law for a second-rank tensor $T_{ij}$, show that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are scalars, i.e. remain the same in all Cartesian coordinate systems. If $T_{ij}$ is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are the roots of the cubic equation

$$\lambda^3 - \alpha\lambda^2 + \frac{1}{2}(\alpha^2 - \beta)\lambda - \frac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$ 

4. If $u_i(\mathbf{x})$ is a vector field, show that $\partial u_i/\partial x_j$ transforms as a second rank tensor field. If $\sigma_{ij}(\mathbf{x})$ is a tensor field, show that $\partial\sigma_{ij}/\partial x_j$ transforms as a vector field.

5. The electric field $\mathbf{E}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$ satisfy Maxwell’s equations with zero charge and current. Show that the Poynting vector $\mathbf{P} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ satisfies the conservation law

$$\frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{\mu_0} \left( \frac{1}{2} \delta_{ij}(E^2 + c^2B^2) - (E_iE_j + c^2B_iB_j) \right)$$

where $c^2 = 1/\mu_0\epsilon_0$. If the component $P_i$ is the momentum density in the $x^i$ direction stored in the electric and magnetic fields, what is the interpretation of $T_{ij}$?
6. The velocity field \( \mathbf{u}(\mathbf{x}, t) \) of an inviscid compressible gas obeys

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{and} \quad \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p
\]

where \( \rho(\mathbf{x}, t) \) is the density and \( p(\mathbf{x}, t) \) is the pressure. Show that

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u_j u_j \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho u_j u_j u_i + pu_i \right) = p \nabla \cdot \mathbf{u} \quad \text{and} \quad \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial t_{ij}}{\partial x_j} = 0
\]

for a suitable symmetric tensor \( t_{ij} \), to be determined.

7. By first decomposing into symmetric and anti-symmetric parts, show that an arbitrary second rank tensor \( T_{ij} \) can be written in the form

\[ T_{ij} = \alpha \delta_{ij} + \epsilon_{ijk} \omega_k + B_{ij} \]

where \( \alpha \) is a scalar, \( \omega_k \) a vector and \( B_{ij} \) a symmetric second rank tensor satisfying \( B_{ii} = 0 \).

   (i) A tensor of rank 3 satisfies \( T_{ijk} = T_{jik} \) and \( T_{ijk} = -T_{ikj} \). Show that \( T_{ijk} = 0 \).
   (ii) A tensor of rank 4 satisfies \( T_{ijkl} = -T_{ijlk} \) and \( T_{ijij} = 0 \). Show that

\[ T_{ijkl} = \epsilon_{ijp} \epsilon_{klq} S_{pq} \quad \text{where} \quad S_{pq} = -T_{rqp}. \]

8. A cuboid of uniform density and mass \( M \) has sides of length \( 2a, 2b \) and \( 2c \). Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

   A cube of sides of length \( 2a \) has uniform charge density, mass \( M \), and is rotating with angular velocity \( \mathbf{\omega} \) about an axis which passes through its center and through a pair of opposite vertices. What is its angular momentum?

9. Evaluate the following integrals, where \( \gamma > 0 \) and \( r^2 = x_p x_p \):

   (i) \( \int_{\mathbb{R}^3} r^{-3} e^{-\gamma r^2} x_i x_j \, dV \),
   (ii) \( \int_{\mathbb{R}^3} r^{-5} e^{-\gamma r^2} x_i x_j x_k \, dV \).

10. A tensor has components \( T_{ij} \) with respect to a given Cartesian coordinate system \( \{x_i\} \). If the tensor is invariant under arbitrary rotations about the \( x_3 \)-axis, show that it must have the form

\[
T_{ij} = \begin{pmatrix}
\alpha & \omega & 0 \\
-\omega & \alpha & 0 \\
0 & 0 & \beta
\end{pmatrix}
\]
11. In the theory of linear elasticity, the symmetric stress tensor $\sigma_{ij}$ depends on the symmetric strain tensor $e_{kl}$ through the equation $\sigma_{ij} = C_{ijkl}e_{kl}$. Explain why $C_{ijkl}$ must be a fourth rank tensor with $C_{ijkl} = C_{jikl}$ and why it may be chosen such that $C_{ijkl} = C_{ijlk}$. For an isotropic medium, use the most general possible form for $C_{ijkl}$ (which you may quote) to show that

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where $\lambda$ and $\mu$ are scalars. Invert this equation to express $e_{ij}$ in terms of $\sigma_{ij}$, assuming $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of $\sigma_{ij}$ and $e_{ij}$ coincide.

The elastic energy density resulting from a deformation of the medium is

$$E = \frac{1}{2} e_{ij} \sigma_{ij}.$$  

Show that $E$ is strictly positive for any non-zero strain $e_{ij}$ provided $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$.

12*. The totally anti-symmetric tensor of rank $n$ is defined by

$$\epsilon_{i_1 i_2 \cdots i_n} = \left\{ \begin{array}{ll} +1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an even permutation of } (1, 2, \ldots, n), \\ -1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an odd permutation of } (1, 2, \ldots, n), \\ 0, & \text{otherwise}. \end{array} \right.$$  

Show that $\epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} = n!$.

How might you get a computer to compute $\epsilon_{i_1 \cdots i_n}$ for a given permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$? For instance, computing the determinant of the relevant permutation matrix is one option, but is quite computationally complex. Can you do better?

13*. Let $T_{ij \cdots k}$ be a tensor of rank $m$ in $\mathbb{R}^n$. How many independent components does $T_{ij \cdots k}$ have if it is (a) totally antisymmetric; or (b) totally symmetric?