1. The current \( J \) due to an electric field \( E \) is given by \( J_i = \sigma_{ij} E_j \), where \( \sigma_{ij} \) is the conductivity tensor. In a given Cartesian coordinate system

\[
\sigma_{ij} = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}
\]

Show that there is a direction along which no current flows, and find the direction(s) along which the current is largest, for an electric field of fixed magnitude.

2. Given the vectors \( u = (1, 0, 1) \), \( v = (0, 1, -1) \) and \( w = (1, 1, 0) \), find all components of the second-rank and third-rank tensors defined by

\[
T_{ij} = u_i v_j + v_i w_j; \quad S_{ijk} = u_i v_j w_k - v_i u_j w_k + w_i v_j u_k - w_i u_j v_k.
\]

3. Use the transformation law for a second-rank tensor \( T_{ij} \), show that the quantities

\[
\alpha = T_{ii}, \quad \beta = T_{ij} T_{ji}, \quad \gamma = T_{ij} T_{jk} T_{ki}
\]

are scalars, i.e. remain the same in all Cartesian coordinate systems. If \( T_{ij} \) is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are the roots of the cubic equation

\[
\lambda^3 - \alpha \lambda^2 + \frac{1}{2} (\alpha^2 - \beta) \lambda - \frac{1}{6} (\alpha^3 - 3 \alpha \beta + 2 \gamma) = 0.
\]

4. If \( u_i(x) \) is a vector field, show that \( \partial u_i/\partial x_j \) transforms as a second rank tensor field. If \( \sigma_{ij}(x) \) is a tensor field, show that \( \partial \sigma_{ij}/\partial x_j \) transforms as a vector field.

5. The electric field \( E(x, t) \) and magnetic field \( B(x, t) \) satisfy Maxwell’s equations with zero charge and current. Show that the Poynting vector \( P = \frac{1}{\mu_0} E \times B \) satisfies the conservation law

\[
\frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{\mu_0} \left( \frac{1}{2} \delta_{ij} (E^2 + c^2 B^2) - (E_i E_j + c^2 B_i B_j) \right)
\]

where \( c^2 = 1/\mu_0 \epsilon_0 \). If the component \( P_i \) is the momentum density in the \( x_i \) direction stored in the electric and magnetic fields, what is the interpretation of \( T_{ij} \)?
6. The velocity field \( u(x, t) \) of an inviscid compressible gas obeys
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad \text{and} \quad \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = -\nabla P
\]
where \( \rho(x, t) \) is the density and \( P(x, t) \) is the pressure. Show that
\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u_j u_j \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho u_j u_i + Pu_i \right) = P \nabla \cdot u \quad \text{and} \quad \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial t_{ij}}{\partial x_j} = 0
\]
for a suitable symmetric tensor \( t_{ij} \), to be determined.

7. By first decomposing into symmetric and anti-symmetric parts, show that an arbitrary second rank tensor \( T_{ij} \) can be written in the form
\[
T_{ij} = \alpha \delta_{ij} + \epsilon_{ijk} \omega_k + B_{ij}
\]
where \( \alpha \) is a scalar, \( \omega_k \) a vector and \( B_{ij} \) a symmetric second rank tensor satisfying \( B_{ii} = 0 \).

(i) A tensor of rank 3 satisfies \( T_{ijk} = T_{jik} \) and \( T_{ijk} = -T_{ikj} \). Show that \( T_{ijk} = 0 \).

(ii) A tensor of rank 4 satisfies \( T_{jikl} = -T_{ijkl} = T_{ijlk} \) and \( T_{ijij} = 0 \). Show that
\[
T_{ijkl} = \epsilon_{ijp} \epsilon_{klq} S_{pq} \quad \text{where} \quad S_{pq} = -T_{rqrp}.
\]

8. A cuboid of uniform density and mass \( M \) has sides of length \( 2a, 2b \) and \( 2c \). Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

A cube of sides of length \( 2a \) has uniform charge density, mass \( M \), and is rotating with angular velocity \( \omega \) about an axis which passes through its center and through a pair of opposite vertices. What is its angular momentum?

9. Evaluate the following integrals, where \( \gamma > 0 \) and \( r^2 = x_p x_p \):

(i) \[ \int_{\mathbb{R}^3} r^{-3} e^{-\gamma r^2} x_i x_j \, dV, \]
(ii) \[ \int_{\mathbb{R}^3} r^{-5} e^{-\gamma r^2} x_i x_j x_k \, dV. \]

10. A tensor has components \( T_{ij} \) with respect to a given Cartesian coordinate system \( \{x_i\} \). If the tensor is invariant under arbitrary rotations about the \( x_3 \)-axis, show that it must have the form
\[
T_{ij} = \begin{pmatrix}
\alpha & \omega & 0 \\
-\omega & \alpha & 0 \\
0 & 0 & \beta
\end{pmatrix}
\]
11. In the theory of linear elasticity, the symmetric stress tensor \( \sigma_{ij} \) depends on the symmetric strain tensor \( e_{kl} \) through the equation \( \sigma_{ij} = C_{ijkl} e_{kl} \). Explain why \( C_{ijkl} \) must be a fourth rank tensor, assuming that \( C_{ijkl} = C_{ijlk} \). For an isotropic medium, use the most general possible form for \( C_{ijkl} \) (which you may quote) to show that

\[
\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},
\]

where \( \lambda \) and \( \mu \) are scalars. Invert this equation to express \( e_{ij} \) in terms of \( \sigma_{ij} \), assuming \( \mu \neq 0 \) and \( 3\lambda \neq -2\mu \). Explain why the principal axes of \( \sigma_{ij} \) and \( e_{ij} \) coincide.

The elastic energy density resulting from a deformation of the medium is \( E = \frac{1}{2} e_{ij} \sigma_{ij} \). Show that \( E \) is strictly positive for any non-zero strain \( e_{ij} \) provided \( \mu > 0 \) and \( \lambda > -\frac{2}{3}\mu \).

12*. The totally anti-symmetric tensor of rank \( n \) is defined by

\[
\epsilon_{i_1 i_2 \cdots i_n} = \begin{cases} +1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an even permutation of } (1, 2, \ldots, n), \\ -1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an odd permutation of } (1, 2, \ldots, n), \\ 0, & \text{otherwise.} \end{cases}
\]

Show that \( \epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} = n! \).

How might you get a computer to compute \( \epsilon_{i_1 \cdots i_n} \) for a given permutation \( (i_1, \ldots, i_n) \) of \( (1, \ldots, n) \)? For instance, computing the determinant of the relevant permutation matrix is one option, but is quite computationally complex. Can you do better?

13*. Let \( T_{ij \cdots k} \) be a tensor of rank \( m \) in \( \mathbb{R}^n \). How many independent components does \( T_{ij \cdots k} \) have if it is (a) totally antisymmetric; or (b) totally symmetric?