1 The current $J_i$ due to an electric field $E_i$ is given by $J_i = \sigma_{ij}E_j$, where $\sigma_{ij}$ is the conductivity tensor. In a certain coordinate system,
\[
(\sigma_{ij}) = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}.
\]

Show that there is a direction along which no current flows, and find the direction(s) along which the current flow is largest, for an electric field of fixed magnitude.

2 Given vectors $u = (1, 0, 1)$, $v = (0, 1, -1)$ and $w = (1, 1, 0)$, find all components of the second-rank and third-rank tensors defined by
\[
T_{ij} = u_i v_j + v_i w_j ; \quad S_{ijk} = u_i v_j w_k - v_i u_j w_k + w_i u_j v_k - u_i v_j w_k .
\]

3 Using the transformation law for a second-rank tensor $T_{ij}$, show that the quantities
\[
\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \gamma = T_{ij}T_{jk}T_{ki}
\]
are the same in all Cartesian coordinate systems. If $T_{ij}$ is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are roots of the cubic equation
\[
\lambda^3 - \alpha\lambda^2 + \frac{1}{2}(\alpha^2 - \beta)\lambda - \frac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.
\]

4 If $u_i(x)$ is a vector field, show that $\partial u_i / \partial x_j$ transforms as a second-rank tensor.

If $\sigma_{ij}(x)$ is a second-rank tensor field, show that $\partial \sigma_{ij} / \partial x_j$ transforms as a vector.

5 The fields $E(x, t)$ and $B(x, t)$ obey Maxwell’s equations with zero charge and current. Show that the Poynting vector $P = \mu_0^{-1} E \times B$ satisfies
\[
\frac{1}{c^2} \frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{2} \epsilon_0 \delta_{ij} (E_k E_k + c^2 B_k B_k) - \epsilon_0 (E_i E_j + c^2 B_i B_j).
\]

6 The velocity field $u(x, t)$ of an inviscid compressible gas obeys
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad \text{and} \quad \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = -\nabla p
\]
where $\rho(x, t)$ is the density and $p(x, t)$ is the pressure. Show that
\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho u^2 u_i + p u_i \right) = p \nabla \cdot u \quad \text{and} \quad \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (t_{ij}) = 0
\]
for a suitable symmetric tensor $t_{ij}$, to be determined.
The components of a second-rank tensor are given by a matrix $A$. Show that $Ax = \alpha x + \omega \times x + Bx$ for all $x$, for some scalar $\alpha$, vector $\omega$, and symmetric traceless matrix $B$. Find $\alpha$, $\omega$ and $B$ when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$ 

8 (a) A tensor of rank 3 satisfies $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{ikj}$. Show that $T_{ijk} = 0$.

(b) A tensor of rank 4 satisfies $T_{jik\ell} = -T_{ijk\ell} = T_{ij\ell k}$ and $T_{ijij} = 0$. Show that $T_{ijk\ell} = \varepsilon_{ijp} \varepsilon_{k\ell q} S_{pq}$, where $S_{pq} = -T_{rqrp}$.

9 A cuboid of uniform density and mass $M$ has sides of lengths $2a$, $2b$ and $2c$. Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

A cube with sides of length $2a$ has uniform density, mass $M$, and is rotating with angular velocity $\omega$ about an axis which passes through its centre and through a pair of opposite vertices. What is its angular momentum?

10 Evaluate the following integrals over all space, where $\gamma > 0$ and $r^2 = x_p x_p$:

(i) $\int r^{-3} e^{-\gamma r^2} x_i x_j dV$;  
(ii) $\int r^{-5} e^{-\gamma r^2} x_i x_j x_k dV$.

11 A tensor has components $T_{ij}$ with respect to Cartesian coordinates $x_i$. If the tensor is invariant under arbitrary rotations around the $x_3$-axis, show that it must have the form

$$(T_{ij}) = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$ 

12 In linear elasticity, the symmetric second-rank stress tensor $\sigma_{ij}$ depends on the symmetric second-rank strain tensor $e_{kl}$ according to $\sigma_{ij} = c_{ijkl} e_{kl}$. Explain why $c_{ijkl}$ must be a fourth-rank tensor, assuming $e_{ijkl} = c_{ijkl}$. For an isotropic medium, use the most general possible form for $c_{ijkl}$ (which you may quote) to show that

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$ 

where $\lambda$ and $\mu$ are scalars.

Invert this equation to express $e_{ij}$ in terms of $\sigma_{ij}$, assuming $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of $\sigma_{ij}$ and $e_{ij}$ coincide.

The elastic energy density resulting from a deformation of the medium is $E = \frac{1}{2} e_{ij} \sigma_{ij}$. Show that $E$ is strictly positive for any non-zero strain $e_{ij}$ provided $\mu > 0$ and $\lambda > -2\mu/3$.

Comments to: B.C.Allanach@damtp.cam.ac.uk