

Comments and corrections to [acla2@damtp.cam.ac.uk](mailto:acla2@damtp.cam.ac.uk). Sheet with commentary available to supervisors.

1. The current  $J_i$  due to an electric field  $E_i$  is given by  $J_i = \sigma_{ij}E_j$ , where  $\sigma_{ij}$  is the conductivity tensor. In a given Cartesian coordinate system,

$$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current is largest, for an electric field of fixed magnitude.

2. Given the vectors  $\mathbf{u} = (1, 0, 1)$ ,  $\mathbf{v} = (0, 1, -1)$  and  $\mathbf{w} = (1, 1, 0)$ , find all components of the second-rank and third-rank tensors defined by

$$T_{ij} = u_i v_j + v_i w_j; \quad S_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k - w_i v_j u_k + w_i u_j v_k - u_i w_j v_k.$$

3. Use the transformation law for a second-rank tensor  $T_{ij}$ , show that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are scalars, i.e. remain the same in all Cartesian coordinate systems. If  $T_{ij}$  is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are the roots of the cubic equation

$$\lambda^3 - \alpha\lambda^2 + \frac{1}{2}(\alpha^2 - \beta)\lambda - \frac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

4. If  $u_i(\mathbf{x})$  is a vector field, show that  $\partial u_i / \partial x_j$  transforms as a second rank tensor field. If  $\sigma_{ij}(\mathbf{x})$  is a tensor field, show that  $\partial \sigma_{ij} / \partial x_j$  transforms as a vector field.

5. The vector fields  $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$  and  $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$  satisfy Maxwell's equations with zero charge and current. Show that the Poynting vector  $\mathbf{P} = \mu_0^{-1} \mathbf{E} \times \mathbf{B}$  satisfies the conservation law

$$\frac{1}{c^2} \frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{2} \epsilon_0 \delta_{ij} (E_k E_k + c^2 B_k B_k) - \epsilon_0 (E_i E_j + c^2 B_i B_j).$$

6. The velocity field  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$  of an inviscid compressible gas obeys

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{and} \quad \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p,$$

where  $\rho = \rho(\mathbf{x}, t)$  is the density and  $p = p(\mathbf{x}, t)$  is the pressure. Show that

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u_j u_j \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho u_j u_j u_i + p u_i \right) = p \nabla \cdot \mathbf{u} \quad \text{and} \quad \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (t_{ij}) = 0$$

for a suitable symmetric tensor  $t_{ij}$ , to be determined.

7. By first decomposing into symmetric and anti-symmetric parts, show that an arbitrary second rank tensor  $T_{ij}$  can be written in the form

$$T_{ij} = \alpha \delta_{ij} + \epsilon_{ijk} \omega_k + B_{ij}$$

where  $\alpha$  is a scalar,  $\omega_k$  a vector and  $B_{ij}$  a symmetric second rank tensor satisfying  $B_{ii} = 0$ .

8. (i) A tensor of rank 3 satisfies  $T_{ijk} = T_{jik}$  and  $T_{ijk} = -T_{ikj}$ . Show that  $T_{ijk} = 0$ .

(ii) A tensor of rank 4 satisfies  $T_{ijkl} = -T_{jikl} = T_{ijlk}$  and  $T_{ijij} = 0$ . Show that

$$T_{ijkl} = \epsilon_{ijp} \epsilon_{klq} S_{pq} \quad \text{where} \quad S_{pq} = -T_{rqrp}.$$

9. A cuboid of uniform density and mass  $M$  has sides of length  $2a$ ,  $2b$  and  $2c$ . Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

A cube of sides of length  $2a$  has uniform charge density, mass  $M$ , and is rotating with angular velocity  $\boldsymbol{\omega}$  about an axis which passes through its center and through a pair of opposite vertices. What is its angular momentum?

10. Evaluate the following integrals, where  $\gamma > 0$  and  $r^2 = x_p x_p$ :

$$(i) \int_{\mathbf{R}^3} r^{-3} e^{-\gamma r^2} x_i x_j dV, \quad (ii) \int_{\mathbf{R}^3} r^{-5} e^{-\gamma r^2} x_i x_j x_k dV.$$

11. A tensor has components  $T_{ij}$  with respect to a given Cartesian coordinate system  $\{x_i\}$ . If the tensor is invariant under arbitrary rotations about the  $x_3$ -axis, show that it must have the form

$$(T_{ij}) = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

12. In linear elasticity, the symmetric second-rank stress tensor  $\sigma_{ij}$  depends on the symmetric second-rank strain tensor  $e_{kl}$  according to  $\sigma_{ij} = c_{ijkl} e_{kl}$ . Explain why  $c_{ijkl}$  must be a fourth rank tensor, assuming  $c_{ijkl} = c_{ijlk}$ . For an isotropic medium, use the most general possible form for  $c_{ijkl}$  (which you may quote) to show that

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where  $\lambda$  and  $\mu$  are scalars. Invert this equation to express  $e_{ij}$  in terms of  $\sigma_{ij}$ , assuming  $\mu \neq 0$  and  $3\lambda \neq -2\mu$ . Explain why the principal axes of  $\sigma_{ij}$  and  $e_{ij}$  coincide.

The elastic energy density resulting from a deformation of the medium is  $E = \frac{1}{2} e_{ij} \sigma_{ij}$ . Show that  $E$  is strictly positive for any non-zero strain  $e_{ij}$  provided  $\mu > 0$  and  $\lambda > -\frac{2}{3}\mu$ .

## Additional problems

*These questions should **not** be attempted at the expense of earlier ones.*

13. The Levi-Civita tensor of rank  $n$  is defined by

$$\epsilon_{i_1 i_2 \dots i_n} = \begin{cases} +1, & \text{if } (i_1, i_2, \dots, i_n) \text{ is an even permutation of } (1, 2, \dots, n), \\ -1, & \text{if } (i_1, i_2, \dots, i_n) \text{ is an odd permutation of } (1, 2, \dots, n), \\ 0, & \text{otherwise.} \end{cases}$$

This is the  $n$ -dimensional analogue of  $\epsilon_{ijk}$ . Show that  $\epsilon_{i_1 i_2 \dots i_n} \epsilon_{i_1 i_2 \dots i_n} = n!$ . How might you get a computer to compute  $\epsilon_{i_1 \dots i_n}$  for a given permutation  $(i_1, \dots, i_n)$  of  $(1, \dots, n)$ ? For instance, computing the determinant of the relevant permutation matrix is one option, but is quite computationally complex. Can you do better?

14. Let  $(x_0, x_1, x_2, x_3) \equiv (t, x_1, x_2, x_3)$  be coordinates for Minkowski space  $\mathbf{R}^{1+3}$ . Show that the two Maxwell equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

are equivalent to

$$\frac{\partial F_{ab}}{\partial x_c} + \frac{\partial F_{bc}}{\partial x_a} + \frac{\partial F_{ca}}{\partial x_b} = 0, \quad \text{where } (F_{ab}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

15. Let  $T_{ij\dots k}$  be an  $m$ th rank tensor on  $\mathbf{R}^n$ , so each of the indices can take on  $n$  different values. How many independent components does  $T_{ij\dots k}$  have if it is (a) totally antisymmetric; or (b) totally symmetric?