Line, Area and Volume Elements in Cylindrical Polars

$$\mathbf{r}(\rho,\phi,z) \;=\; \rho\cos\phi\,\mathbf{i} \,+\, \rho\sin\phi\,\mathbf{j} \,+\, z\,\mathbf{k}$$

Line element:

$$\mathrm{d}\mathbf{r} \;=\; h_\rho\,\mathbf{e}_\rho\,\mathrm{d}\rho \,+\, h_\phi\,\mathbf{e}_\phi\,\mathrm{d}\phi \,+\, h_z\,\mathbf{e}_z\,\mathrm{d}z$$

$$\begin{aligned} h_\rho &= 1 & \mathbf{e}_\rho &= \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \\ h_\phi &= \rho & \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \\ h_z &= 1 & \mathbf{e}_z &= \mathbf{k} \end{aligned}$$

Area elements:

$$\mathrm{d}\mathbf{S} \;=\; \begin{cases} \mathbf{e}_\rho\,\rho\,\mathrm{d}\phi\,\mathrm{d}z & (\rho \text{ const}) \\ \mathbf{e}_\phi\,\mathrm{d}\rho\,\mathrm{d}z & (\phi \text{ const}) \\ \mathbf{e}_z\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi & (z \text{ const}) \end{cases}$$

Volume element:

$$\mathrm{d}V \;=\; \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$$

Differential Operators in Cylindrical Polars

$$\nabla \;=\; \frac{1}{h_\rho}\,\mathbf{e}_\rho\,\frac{\partial}{\partial\rho} \,+\, \frac{1}{h_\phi}\,\mathbf{e}_\phi\,\frac{\partial}{\partial\phi} \,+\, \frac{1}{h_z}\,\mathbf{e}_z\,\frac{\partial}{\partial z}$$

Gradient:

$$\nabla f \;=\; \frac{\partial f}{\partial\rho}\mathbf{e}_\rho \,+\, \frac{1}{\rho}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi \,+\, \frac{\partial f}{\partial z}\mathbf{e}_z$$

Curl:

$$\nabla\times\mathbf{F} \;=\; \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\,\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

Divergence:

$$\nabla\cdot\mathbf{F} \;=\; \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) \,+\, \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} \,+\, \frac{\partial F_z}{\partial z}$$

Laplacian:

$$\nabla^2 f \;=\; \frac{\partial^2 f}{\partial\rho^2} \,+\, \frac{1}{\rho}\frac{\partial f}{\partial\rho} \,+\, \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} \,+\, \frac{\partial^2 f}{\partial z^2}$$

for a scalar field $f(\mathbf{r})$ and a vector field $\mathbf{F}(\mathbf{r}) \;=\; F_\rho\mathbf{e}_\rho \,+\, F_\phi\mathbf{e}_\phi \,+\, F_z\mathbf{e}_z$

Line, Area and Volume Elements in Spherical Polars

$$\mathbf{r}(r,\theta,\phi) \;=\; r\sin\theta\,\cos\phi\,\mathbf{i} \,+\, r\sin\theta\,\sin\phi\,\mathbf{j} \,+\, r\cos\theta\,\mathbf{k}$$

Line element:

$$\mathrm{d}\mathbf{r} \;=\; h_r\,\mathbf{e}_r\,\mathrm{d}r \,+\, h_\theta\,\mathbf{e}_\theta\,\mathrm{d}\theta \,+\, h_\phi\,\mathbf{e}_\phi\,\mathrm{d}\phi$$

$$\begin{aligned} h_r &= 1 & \mathbf{e}_r &= \sin\theta\,\cos\phi\,\mathbf{i} + \sin\theta\,\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \\ h_\theta &= r & \mathbf{e}_\theta &= \cos\theta\,\cos\phi\,\mathbf{i} + \cos\theta\,\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\ h_\phi &= r\sin\theta & \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \end{aligned}$$

Area elements:

$$\mathrm{d}\mathbf{S} \;=\; \begin{cases} \mathbf{e}_r\,r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi & (r\text{ const}) \\ \mathbf{e}_\theta\,r\sin\theta\,\mathrm{d}r\,\mathrm{d}\phi & (\theta\text{ const}) \\ \mathbf{e}_\phi\,r\,\mathrm{d}r\,\mathrm{d}\theta & (\phi\text{ const}) \end{cases}$$

Volume element:

$$\mathrm{d}V \;=\; r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Differential Operators in Spherical Polars

$$\nabla \;=\; \frac{1}{h_r}\,\mathbf{e}_r\,\frac{\partial}{\partial r} \,+\, \frac{1}{h_\theta}\,\mathbf{e}_\theta\,\frac{\partial}{\partial\theta} \,+\, \frac{1}{h_\phi}\,\mathbf{e}_\phi\,\frac{\partial}{\partial\phi}$$

Gradient:

$$\nabla f \;=\; \frac{\partial f}{\partial r}\mathbf{e}_r \,+\, \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta \,+\, \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi$$

Curl:

$$\nabla\times\mathbf{F} \;=\; \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & r\,F_\theta & r\sin\theta F_\phi \end{vmatrix}$$

Divergence:

$$\nabla\cdot\mathbf{F} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) \,+\, \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) \,+\, \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

Laplacian:

$$\nabla^2 f \;=\; \frac{\partial^2 f}{\partial r^2} \,+\, \frac{2}{r}\frac{\partial f}{\partial r} \,+\, \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial f}{\partial\theta}\Big) \,+\, \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

for a scalar field $f(\mathbf{r})$ and a vector field $\mathbf{F}(\mathbf{r}) = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$