Mathematical Tripos Part IA | **Lent Term 2013**
Vector Calculus | **Dr. J.M. Evans**

Notes on Calculus and Coordinates

Single Variable Calculus $\mathbf{R} \to \mathbf{R}$

• A function $f : \mathbf{R} \to \mathbf{R}$ is *differentiable* at x if

$$f(x+h) \;=\; f(x) \;+\; mh \;+\; \mathrm{o}(h) \qquad \text{as} \qquad h \to 0 \;,$$

for some real number m. This number is the *derivative* of f at x:

$$m \;=\; f'(x) \;=\; \lim_{h\to 0} \frac{1}{h} \left\{ f(x+h) - f(x) \right\} \;.$$

The definition says that f can be approximated in a neighbourhood of x by a line (the tangent to the graph of f at x, with gradient m). This is a *local linearisation* of f.
[Recall little-o notation: $a(h) = b(h) + \mathrm{o}(h)$ as $h \to 0$ iff $(\,a(h) - b(h)\,)/h \;\to\; 0$ as $h \to 0$.]

• Setting $y = f(x)$ and $m = \mathrm{d}y/\mathrm{d}x$, the statement that the function is differentiable becomes

$$\delta y \;=\; \frac{\mathrm{d}y}{\mathrm{d}x}\,\delta x \;+\; \mathrm{o}(\delta x) \quad \text{as} \quad \delta x \to 0 \;, \qquad \text{or} \qquad \mathrm{d}y \;=\; \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \;.$$

The use of *differentials* allows a convenient abbreviation of the first equation, in which the o-term and limit are suppressed.

• A function $f : \mathbf{R} \to \mathbf{R}$ is *smooth* if it can be differentiated any number of times, so that $f'(x)$, $f''(x)$, … , all exist. The functions we deal with will be smooth except where things go wrong in some obvious way, e.g. $y = 1/x$ is smooth except at $x = 0$, where the function is not defined. We restrict definitions and results to some appropriate subset of $\mathbf{R}$ whenever necessary.

• *Taylor's Theorem* for a smooth function f states:

$$f(x+h) \;=\; f(x) \;+\; f'(x)h \;+\; \frac{1}{2}f''(x)h^2 \;+\; \ldots \;+\; \frac{1}{k!}f^{(k)}(x)h^k \;+\; \mathrm{O}(h^{k+1})$$

for any k, and for h in a suitable range [e.g. x and $x{+}h$ in any closed interval contained within an open interval on which f is smooth]. This does *not* imply that f has a convergent power series expansion, in general; nevertheless, this will often be the case for the functions we meet.

• If $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are smooth functions, then $f \circ g : \mathbf{R} \to \mathbf{R}$ is also smooth, and its derivative at a point u is given by the *Chain Rule*:

$$(f \circ g)'(u) \;=\; f'(g(u))\, g'(u) \;.$$

Setting $y = f(x)$ and $x = g(u)$, the Chain Rule can also be expressed

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}u} \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}u} = \frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}x}$$

where the differential operators act on any function which depends on u only through x.

- The *Riemann integral* of a smooth function F over an interval $a \le x \le b$ is the limit of a sum

$$\int_a^b F(x)\,\mathrm{d}x \;=\; \lim_{\ell\to 0} \sum_I F(x_I^*)\,\delta x_I \ ,$$

which can be defined as follows. Given any ℓ, partition the interval $[a,b]$ into N segments, labelled by I, so that each segment is of length $\delta x_I \le \ell$, and choose any points x_I^* in each segment. As $\ell \to 0$ (implying $N \to \infty$) the partition into segments becomes finer and finer. The limit is independent of all the choices made when F is smooth (or with much weaker assumptions, in fact).

- *Fundamental Theorem of Calculus.* If $F = f'$ for some smooth function f then

$$\int_a^b F(x)\mathrm{d}x \;=\; f(b) - f(a)\ .$$

Understanding how this generalises to higher dimensions is one of our main objectives.

Many Variable Calculus $\mathbf{R}^n \to \mathbf{R}$

- A function $f : \mathbf{R}^n \to \mathbf{R}$ is *differentiable* at x_ℓ (a point in $\mathbf{R}^n$ with these coordinates) iff

$$f(x_\ell + h_\ell) \;=\; f(x_\ell) \,+\, m_i h_i \,+\, \mathrm{o}(h) \qquad \text{as} \qquad h = (h_i h_i)^{1/2} \to 0\ ,$$

for some real numbers m_i. These are the *partial derivatives* of f at x_ℓ:

$$m_i \;=\; \frac{\partial f}{\partial x_i}\ ,$$

with m_i calculated by keeping x_j fixed for $j \neq i$.

[*A subtle point*: The definition appears to depend on a notion of length, h, for the change in coordinates h_i. In fact we could make other choices, e.g. $h = \sum_i |h_i|$ or $h = \max(h_i)$, and it would make no difference to whether the function is differentiable.]

- A function $f : \mathbf{R}^n \to \mathbf{R}$ is *smooth* if successive partial derivatives exist to all orders:

$$\frac{\partial f}{\partial x_i}\ ,\quad \frac{\partial^2 f}{\partial x_i \partial x_j}\ ,\quad \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\ ,\ \ldots\ .$$

The order in which the derivatives are taken is then unimportant, so partial derivatives are totally symmetric, e.g.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \;=\; \frac{\partial^2 f}{\partial x_j \partial x_i}\ .$$

- *Taylor's Theorem* for a smooth function of n variables states:

$$f(x_\ell + h_\ell) \;=\; f(x_\ell) \,+\, \frac{\partial f}{\partial x_i} h_i \,+\, \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j \,+\, \ldots \,+\, \frac{1}{k!}\frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}} h_{i_1} \ldots h_{i_k} \,+\, \mathrm{O}(h^{k+1})$$

with appropriate ranges for each h_i.

Notes on Forces, Fields and Potentials

Newton's Law and Coulomb's Law

• *Newton's Second Law of Motion* governs the behaviour of a particle of mass m, position $\mathbf{r}(t)$, subject to a force $\mathbf{F}(\mathbf{r})$:

$$m\,\ddot{\mathbf{r}}(t) \,=\, \mathbf{F}(\mathbf{r}) \;.$$

Given $\mathbf{F}(\mathbf{r})$, this equation can be solved (in principle) to find $\mathbf{r}(t)$ subject to suitable initial conditions (e.g. specifying $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$).

• The gravitational force between two particles depends on their masses (SI unit kg). *Newton's Law of Gravitation* states that the force on a particle of mass m at position $\mathbf{r}$, due to a particle of mass M at the origin, is:

$$\mathbf{F}(\mathbf{r}) \,=\, -\,G\,\frac{Mm}{r^2}\,\hat{\mathbf{r}} \,=\, -\,G\,\frac{Mm}{r^3}\,\mathbf{r}$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$. *Newton's constant* is $G \approx 6.67{\times}10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$. Newton's Law of Gravitation holds independently of the motion of the particles.

• The electromagnetic force between two particles depends on their electric charges (SI unit *Coulomb*, C). The force also depends on how the charges move, in general, and not just on their relative position. In the *electrostatic* case, however, when the charges are at rest, *Coulomb's Law* states that the force on a particle of charge q which is stationary at $\mathbf{r}$, due to a particle of charge Q stationary at the origin, is:

$$\mathbf{F}(\mathbf{r}) \,=\, \frac{1}{4\pi\epsilon_0}\,\frac{Qq}{r^2}\,\hat{\mathbf{r}} \,=\, \frac{1}{4\pi\epsilon_0}\,\frac{Qq}{r^3}\,\mathbf{r}$$

The factor of 4π is conventional, and the strength of the interaction is given by a constant called the *vacuum permittivity*, $\epsilon_0 \approx 8.85{\times}10^{-12}\,\mathrm{C^2\,N^{-1}m^{-2}}$.

• Both forces above obey an *inverse square law* and the formulas are identical, up to an overall constant, if the roles of mass and charge are interchanged. A difference of sign means that all masses attract under gravity, while like charges repel and unlike charges attract.

Fields and Sources

• The total gravitational or electrostatic force $\mathbf{F}(\mathbf{r})$ produced by some general distribution of masses or charges is the sum of the forces due to each individual mass or charge in the distribution. The result is then proportional to the mass m or charge q of any particle on which the force acts; so by setting

$$\mathbf{F}(\mathbf{r}) \,=\, m\mathbf{g}(\mathbf{r}) \qquad \text{or} \qquad \mathbf{F}(\mathbf{r}) \,=\, q\mathbf{E}(\mathbf{r})$$

we can define the *gravitational field* $\mathbf{g}(\mathbf{r})$ (the gravitational force per unit mass) and the *electric field* $\mathbf{E}(\mathbf{r})$ (the electrostatic force per unit charge) which depend just on the distribution producing the force, independent of the particular mass or charge on which it is acting. The fields $\mathbf{g}$ or $\mathbf{E}$ are said to be *generated* by the distribution, which is said to be a *source* for these fields.

• Newton's Law and Coulomb's Law now become the following results for gravitational or electric fields due to a point mass M or point charge Q at the origin:

$$\mathbf{g}(\mathbf{r}) \;=\; -\,GM\,\frac{1}{r^2}\,\hat{\mathbf{r}} \qquad \text{or} \qquad \mathbf{E}(\mathbf{r}) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{1}{r^2}\,\hat{\mathbf{r}}\;.$$

• A special aspect of gravity is that the same mass m appears both in Newton's Law of Gravitation and in Newton's Law of Motion. The gravitational field $\mathbf{g}(\mathbf{r})$ is therefore just the *acceleration due to gravity*. (This equality of inertial and gravitational mass is the first hint of a modified, purely geometrical theory of gravity, *General Relativity*.)

• The electromagnetic force on a moving charge q depends on both the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ produced by all other charges and currents. It is called the *Lorentz Force*:

$$\mathbf{F} \;=\; q(\,\mathbf{E} + \dot{\mathbf{r}}\times\mathbf{B}\,)\;.$$

Potentials and Poisson's Equation

Topics in this section will be covered more fully in the lectures; this is summary.

• Gravitational and electrostatic forces are conservative, so

$$\nabla\times\mathbf{F} = \mathbf{0}\,, \qquad \mathbf{F}(\mathbf{r}) \;=\; -\nabla V\,,$$

where the scalar function $V(\mathbf{r})$ is the potential energy. It follows that the fields $\mathbf{g}$ and $\mathbf{E}$ are also conservative:

$$\nabla\times\mathbf{g} = \mathbf{0}\,, \qquad \nabla\times\mathbf{E} = \mathbf{0}\,,$$
$$\mathbf{g} = -\nabla\varphi\,, \qquad \mathbf{E} = -\nabla\varphi\,,$$

where $\varphi(\mathbf{r})$ is called the *gravitational potential* or the *electrostatic potential*. These scalar fields are the potential energy per unit mass or per unit charge.

• Directly from Newton's Law or Coulomb's Law, the gravitational or electrostatic potential due to a point mass M or point charge Q at the origin is:

$$\varphi(\mathbf{r}) \;=\; -\,GM\,\frac{1}{r} \qquad \text{or} \qquad \varphi(\mathbf{r}) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{1}{r}\;.$$

• A general distribution of mass or charge can be defined by a scalar field $\rho(\mathbf{r})$, the *mass or charge density*. The fields $\mathbf{g}$ or $\mathbf{E}$ are determined from ρ by Gauss's Law; in its differential form, this states:

$$\nabla\cdot\mathbf{g} = -4\pi G\rho \qquad \text{or} \qquad \nabla\cdot\mathbf{E} = \rho/\epsilon_0$$

Expressing this in terms of potentials gives *Poisson's equation*:

$$\nabla^2\varphi = 4\pi G\rho \qquad \text{or} \qquad \nabla^2\varphi = -\rho/\epsilon_0$$

Given a source $\rho(\mathbf{r})$, solving Poisson's equation for $\varphi(\mathbf{r})$ enables us to find the corresponding gravitational or electrostatic field.

• In any region where the source ρ is zero, Poisson's equation reduces to *Laplace's equation*:

$$\nabla^2\varphi = 0\;.$$

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Notes on the Geometry of Curves

• Consider a curve C parametrised by arc-length, with position vector $\mathbf{r}(s)$ and unit tangent vector

$$\mathbf{t}(s) = \mathbf{r}'(s) = \mathrm{d}\mathbf{r}/\mathrm{d}s \ .$$

If $\mathbf{t}$ is constant then C is a straight line, $\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{t}$. In general, $\mathbf{t}'(s)$ will be non-zero and it then specifies a direction perpendicular to C since, by differentiating,

$$\mathbf{t}^2 = 1 \qquad \Rightarrow \qquad \mathbf{t}\cdot\mathbf{t}' = 0 \ .$$

We use this to define a unit vector $\mathbf{n}(s)$, the *principal normal*, orthogonal to $\mathbf{t}(s)$, by setting

$$\mathrm{d}\mathbf{t}/\mathrm{d}s = \kappa\mathbf{n}$$

where $\kappa(s)$ is called the *curvature*. (The direction of $\mathbf{n}$ can be chosen to make $\kappa > 0$, but this is not essential.) The *binormal* is a third unit vector $\mathbf{b}(s)$ defined by

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} \qquad \text{so} \qquad \{\,\mathbf{t},\,\mathbf{n},\,\mathbf{b}\,\} \quad \text{is a right-handed orthonormal basis.}$$

• Now consider how the basis $\{\mathbf{t},\,\mathbf{n},\,\mathbf{b}\}$ varies with s along C. Note that, again by differentiating,

$$\begin{aligned} \mathbf{n}^2 = 1 \qquad &\Rightarrow \qquad \mathbf{n}\cdot\mathbf{n}' = 0 \\ \mathbf{b}^2 = 1 \qquad &\Rightarrow \qquad \mathbf{b}\cdot\mathbf{b}' = 0 \end{aligned}$$

so that, at each point on C, $\mathbf{n}'$ is a linear combination of $\mathbf{t}$ and $\mathbf{b}$, while $\mathbf{b}'$ is a linear combination of $\mathbf{t}$ and $\mathbf{n}$. But the coefficients in these linear combinations are also constrained by:

$$\begin{aligned} \mathbf{n}\cdot\mathbf{t} = 0 \qquad &\Rightarrow \qquad \mathbf{n}\cdot\mathbf{t}' = -\mathbf{t}\cdot\mathbf{n}' = \kappa \\ \mathbf{t}\cdot\mathbf{b} = 0 \qquad &\Rightarrow \qquad \mathbf{t}\cdot\mathbf{b}' = -\mathbf{b}\cdot\mathbf{t}' = 0 \\ \mathbf{b}\cdot\mathbf{n} = 0 \qquad &\Rightarrow \qquad \mathbf{b}\cdot\mathbf{n}' = -\mathbf{n}\cdot\mathbf{b}' = \tau \end{aligned}$$

The results for the (related) inner-products in the first two lines follow immediately from the definition of $\mathbf{n}$ in terms of $\mathbf{t}'$, while the last line then serves to define a new quantity $\tau(s)$, called the *torsion*. Thus, the variation of the basis vectors with s is specified by two scalar functions $\kappa(s)$ and $\tau(s)$, according to the following *Frenet-Serret* equations:

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= -\kappa\mathbf{t} + \tau\mathbf{b} \\ \mathbf{b}' &= -\tau\mathbf{n} \end{aligned}$$

• The curvature $\kappa(s)$ and torsion $\tau(s)$ encode the geometry of the curve C. Some understanding of this is gained by considering a Taylor expansion of the curve around $s = 0$:

$$\mathbf{r}(s) = \mathbf{r}(0) \;+\; s\mathbf{t} \;+\; \tfrac{1}{2}s^2\kappa\mathbf{n} \;+\; \tfrac{1}{6}s^3(-\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}) \;+\; O(s^4)$$

where the expressions for the coefficients follow just from the definitions given above and all are understood to be evaluated at $s = 0$. To first order in s, the curve C is clearly approximated by its tangent line at $\mathbf{r}(0)$. To second order it can be approximated by a circle through $\mathbf{r}(0)$ which has radius $1/\kappa$ and which lies in a plane spanned by $\mathbf{t}$ and $\mathbf{n}$. To third order C leaves this plane at a rate which depends on τ.

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Notes on Tensors, Multilinear Maps and the Quotient Rule

• Consider orthonormal bases and Cartesian coordinates related by

$$\mathbf{e}'_i = R_{ip}\,\mathbf{e}_p \ , \qquad x'_i = R_{ip}\,x_p \qquad \text{where} \qquad R_{ip}\,R_{jp} = R_{qi}\,R_{qj} = \delta_{ij} \ .$$

The components of any vector $\mathbf{v}$ with respect to these bases satisfy

$$\mathbf{v} = v_p\mathbf{e}_p = v'_i\mathbf{e}'_i \qquad \text{where} \qquad v'_i = R_{ip}\,v_p \ , \quad v_p = R_{ip}\,v'_i \ .$$

• A tensor T of rank n is equivalent to a multilinear map from n vectors to a scalar:

$$T(\mathbf{a},\mathbf{b},\ldots,\mathbf{c}) = T_{pq\ldots r}\,a_p\,b_q\ldots c_r = T'_{ij\ldots k}\,a'_i\,b'_j\ldots c'_k \ .$$

Specifying this scalar for any choice of $\mathbf{a},\mathbf{b},\ldots,\mathbf{c}$ determines the components $T_{pq\ldots r}$ or $T'_{ij\ldots k}$ uniquely. Moreover, the expressions above agree iff

$$T_{pq\ldots r}\,(R_{ip}\,a'_i)\,(R_{jq}\,b'_j)\ldots(R_{kr}\,c'_k) = T'_{ij\ldots k}\,a'_i\,b'_j\ldots c'_k$$

and this is true for any vectors (any a'_i, $b'_j, \ldots, c'_k$) iff

$$T'_{ij\ldots k} = R_{ip}\,R_{jq}\ldots R_{kr}\,T_{pq\ldots r} \qquad \text{Tensor Transformation Rule.}$$

• The tensor product of n vectors $\mathbf{u},\mathbf{v},\ldots,\mathbf{w}$ is equivalent to a multilinear map defined by

$$(\mathbf{u}\otimes\mathbf{v}\otimes\ldots\otimes\mathbf{w})(\mathbf{a},\mathbf{b},\ldots,\mathbf{c}) = (\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\ldots(\mathbf{w}\cdot\mathbf{c}) \ .$$

A tensor T of rank n is not, in general, a tensor product of n vectors, but it can always be written as a linear combination of tensor products of basis vectors, with coefficients precisely the tensor components in each basis:

$$T = T_{pq\ldots r}\,\mathbf{e}_p\otimes\mathbf{e}_q\otimes\ldots\otimes\mathbf{e}_r = T'_{ij\ldots k}\,\mathbf{e}'_i\otimes\mathbf{e}'_j\otimes\ldots\otimes\mathbf{e}'_k \ .$$

This can be checked by applying these expressions, as multilinear maps, to any vectors $\mathbf{a},\mathbf{b},\ldots,\mathbf{c}$.

• Consider an array $T_{i\ldots jp\ldots q}$ with $n{+}m$ indices, defined for each choice of basis. Suppose that for any tensor $U_{p\ldots q}$ of rank m,

$$V_{i\ldots j} = T_{i\ldots jp\ldots q}\,U_{p\ldots q}$$

is a tensor of rank n. Then $T_{i\ldots jp\ldots q}$ is, in fact, a tensor. This is the *Quotient Rule* in its general form. The case $m = n = 1$ (and T a matrix) was dealt with in lectures (section 13.1). To prove the general case, note that if $V_{i\ldots j}$ is a tensor then $V_{i\ldots j}\,a_i\ldots b_j$ is a scalar for any vectors $\mathbf{a},\ldots,\mathbf{b}$. Choosing $U_{p\ldots q} = c_p\ldots d_q$ for any vectors $\mathbf{c},\ldots,\mathbf{d}$, we deduce that

$$T_{i\ldots jp\ldots q}\,a_i\ldots b_j\,c_p\ldots d_q$$

is a scalar for any $\mathbf{a},\ldots,\mathbf{b},\mathbf{c},\ldots,\mathbf{d}$. This implies (as above) the tensor transformation rule for T.