Example Sheet 1

[Example Sheet 1 is based on (approximately) Lectures 1-7 of the course.]

1. In one spatial dimension, two frames of reference $S$ and $S'$ have coordinates $(x, t)$ and $(x', t')$ respectively. The coordinates are related by

$$x' = f(x, t) \quad \text{and} \quad t' = t.$$ 

Viewed in frame $S$, a particle follows a trajectory $x = x(t)$. It has velocity $v = dx/dt$ and acceleration $a = d^2x/dt^2$. Viewed in $S'$, the trajectory is $x' = f(x(t), t)$. Using the chain rule, show that the velocity and acceleration of the particle in $S'$ are given by

$$v' = \frac{dx'}{dt'} = v \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}, \quad a' = \frac{d^2x'}{dt'^2} = a \frac{\partial f}{\partial x} + v^2 \frac{\partial^2 f}{\partial x^2} + 2v \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial t \partial x} + \frac{\partial^2 f}{\partial t^2}.$$

Suppose now that both $S$ and $S'$ are inertial frames. Explain why the function $f$ must obey $\partial^2 f/\partial x^2 = \partial^2 f/\partial x \partial t = \partial^2 f/\partial t^2 = 0$. What is the most general form of $f$ with these properties? Interpret this result.

2. A particle of mass $m$ experiences a force field

$$\mathbf{F}(r) = \left(-\frac{a}{r^2} + \frac{2b}{r^3}\right) \hat{r},$$

where $\hat{r} = r/r$ is a unit vector in the radial direction and $a$ and $b$ are positive constants. Show, by finding a potential energy $V(r)$ such that $\mathbf{F} = -\nabla V$, that $\mathbf{F}$ is conservative. (You will need the result $\nabla r = \hat{r}$.)

Sketch $V(r)$ and describe qualitatively the possible motions of the particle moving in the radial direction, considering different initial positions and velocities. If the particle starts at $r = 2b/a$, what is the minimum speed that it must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance $R$, much larger than the Earth’s radius. Treating the Earth as a point of mass $M$, use dimensional analysis to show that the time $T$ taken by the satellite to reach the Earth is given by

$$T = C \left(\frac{R^3}{GM}\right)^{1/2},$$

where $G$ is the gravitational constant and $C$ is a dimensionless constant. (You will need the fact that the acceleration due to the Earth’s gravitational field at a distance $r$ from the centre of the Earth is $GM/r^2$.)

What is the equation of energy conservation for the satellite? By solving this differential equation, show that $C = \pi/2\sqrt{2}$. 
4. A long time ago in a galaxy far, far away, a Death Star was constructed. Its surrounding
force field caused any particle at position \( r \) relative to the centre of the Death Star to
experience an acceleration
\[
\ddot{r} = \lambda \mathbf{r} \times \dot{\mathbf{r}},
\]
where \( \lambda \) is a constant. Show that the particle moves in this force field with constant
speed. Show also that the magnitude of its acceleration is constant.

(a) A particle is projected radially with speed \( v \) from a point \( r = R \hat{r} \) on the surface of
the Death Star. Show that its trajectory is given by
\[
r = (R + vt) \hat{r}.
\]
(b) By considering the second derivative of \( r \cdot r \) show that, for any particle moving in the
force field, the distance \( r \) from the centre of the Death Star is given by
\[
r^2 = v^2(t - t_0)^2 + r_0^2,
\]
where \( t_0 \) and \( r_0 \) are constants and \( v \) is the speed of the particle. Obtain an expression
for \( r \cdot \dot{r} \) and show that \( |\ddot{r}| = \lambda r_0 v \).

5. A particle of mass \( m \), charge \( q \) and position \( x(t) \) moves in both a uniform magnetic
field \( \mathbf{B} \), which points in a horizontal direction, and a uniform gravitational field \( \mathbf{g} \), which
points vertically downwards. Write down the equation of motion and show that it is
invariant under translations \( x \mapsto x + x_0 \). Show that
\[
\dot{x} = \omega x \times \mathbf{n} + gt + a,
\]
where \( \omega = qB/m \) is the gyrofrequency, \( \mathbf{n} \) is a unit vector in the direction of \( \mathbf{B} \), and \( a \) is
a constant vector. Show also that, with a suitable choice of origin, \( a \) can be written in the form \( a = a \mathbf{n} \).

By choosing suitable axes, show that the particle undergoes a helical motion together
with a constant horizontal drift perpendicular to \( \mathbf{B} \).

Suppose that you now wish to eliminate the drift by imposing a uniform electric field \( \mathbf{E} \).
Determine the direction and magnitude of \( \mathbf{E} \).

6. At time \( t = 0 \), an insect of mass \( m \) jumps from a point \( O \) on the ground with velocity
\( \mathbf{v} \), while a wind blows with constant velocity \( \mathbf{u} \). The gravitational acceleration is \( \mathbf{g} \) and
the air exerts a drag force on the insect equal to \( mk \) times the velocity of the wind relative
to the insect.

(a) Show that the path of the insect is given by
\[
x = \left( \mathbf{u} + \frac{\mathbf{g}}{k} \right) t + \frac{1}{k} \left( v - \frac{\mathbf{g}}{k} \right) \left( v - \mathbf{u} - \frac{\mathbf{g}}{k} \right).
\]
(b) In the case where the insect jumps vertically in a horizontal wind, show that the
time $T$ that elapses before it returns to the ground (which is also horizontal) satisfies

$$1 - e^{-kT} = \frac{kT}{1 + \lambda},$$

where $\lambda = kv/g$. Find an expression for the horizontal range $R$ in terms of $\lambda$, $u$ and $T$. (Here $v = |v|$, $g = |g|$ and $u = |u|$.)

7. A ball of mass $m$ moves, under gravity, in a resistive medium that produces a frictional
force of magnitude $kv^2$, where $v$ is the ball’s speed. If the ball is projected vertically
upwards with initial speed $u$, show by dimensional analysis that when the ball returns to
its point of projection, its speed $w$ can be written in the form

$$w = uf(\lambda),$$

where $\lambda = ku^2/mg$.

Integrate the equation of motion to show that $f(\lambda) = (1 + \lambda)^{-1/2}$. [Hint: Thinking about $v$ as a function of time may not be the easiest approach.]. Discuss what happens in the
two extremes $\lambda \gg 1$ and $\lambda \ll 1$.

*Please send any comments and corrections to phh1@cam.ac.uk*