Example Sheet 1

1. In one spatial dimension, two frames of reference $S$ and $S'$ have coordinates $(x, t)$ and $(x', t')$ respectively. The coordinates are related by

$$x' = f(x, t) \quad \text{and} \quad t' = t.$$ 

Viewed in frame $S$, a particle follows a trajectory $x = x(t)$. It has velocity $v = dx/dt$ and acceleration $a = d^2x/dt^2$. Viewed in $S'$, the trajectory is $x' = f(x(t), t)$. Using the chain rule, show that the velocity and acceleration of the particle in $S'$ are given by

$$v' = \frac{dx'}{dt'} = v \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}, \quad a' = \frac{d^2x'}{dt'^2} = \frac{d^2f}{dx^2} + v^2 \frac{\partial^2 f}{\partial x \partial t} + 2v \frac{\partial f}{\partial x} \frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial t^2}.$$ 

Suppose now that both $S$ and $S'$ are inertial frames. Explain why the function $f$ must obey $\partial^2 f/\partial x^2 = \partial^2 f/\partial x \partial t = \partial^2 f/\partial t^2 = 0$. What is the most general form of $f$ with these properties? Interpret this result.

2. A particle of mass $m$ experiences a force field

$$F(r) = \left(\frac{-a}{r^2} + 2b r^3\right) \hat{r},$$

where $\hat{r} = r/r$ is a unit vector in the radial direction and $a$ and $b$ are positive constants. Show, by finding a potential energy $V(r)$ such that $F = -\nabla V$, that $F$ is conservative. (You will need the result $\nabla r = \hat{r}$.)

Sketch $V(r)$ and describe qualitatively the possible motions of the particle moving in the radial direction, considering different initial positions and velocities. If the particle starts at $r = 2b/a$, what is the minimum speed that it must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance $R$, much larger than the Earth’s radius. Treating the Earth as a point of mass $M$, use dimensional analysis to show that the time $T$ taken by the satellite to reach the Earth is given by

$$T = C \left(\frac{R^3}{GM}\right)^{1/2},$$

where $G$ is the gravitational constant and $C$ is a dimensionless constant. (You will need the fact that the acceleration due to the Earth’s gravitational field at a distance $r$ from the centre of the Earth is $GM/r^2$.)

What is the equation of energy conservation for the satellite? By solving this differential equation, show that $C = \pi/2\sqrt{2}$. 

4. A long time ago in a galaxy far, far away, a Death Star was constructed. Its surrounding force field caused any particle at position $r$ relative to the centre of the Death Star to experience an acceleration

$$\ddot{r} = \lambda r \times \dot{r},$$

where $\lambda$ is a constant. Show that the particle moves in this force field with constant speed. Show also that the magnitude of its acceleration is constant.

(a) A particle is projected radially with speed $v$ from a point $r = R\hat{r}$ on the surface of the Death Star. Show that its trajectory is given by

$$r = (R + vt)\hat{r}.$$

(b) By considering the second derivative of $r \cdot r$ show that, for any particle moving in the force field, the distance $r$ from the centre of the Death Star is given by

$$r^2 = v^2(t - t_0)^2 + r_0^2,$$

where $t_0$ and $r_0$ are constants and $v$ is the speed of the particle. Obtain an expression for $r \cdot \dot{r}$ and show that $|\ddot{r}| = \lambda r_0v$.

5. A particle of mass $m$, charge $q$ and position $x(t)$ moves in both a uniform magnetic field $B$, which points in a horizontal direction, and a uniform gravitational field $g$, which points vertically downwards. Write down the equation of motion and show that it is invariant under translations $x \mapsto x + x_0$. Show that

$$\dot{x} = \omega x \times n + gt + a,$$

where $\omega = qB/m$ is the gyrofrequency, $n$ is a unit vector in the direction of $B$, and $a$ is a constant vector. Show also that, with a suitable choice of origin, $a$ can be written in the form $a = an$.

By choosing suitable axes, show that the particle undergoes a helical motion together with a constant horizontal drift perpendicular to $B$.

Suppose that you now wish to eliminate the drift by imposing a uniform electric field $E$. Determine the direction and magnitude of $E$.

6. At time $t = 0$, an insect of mass $m$ jumps from a point $O$ on the ground with velocity $v$, while a wind blows with constant velocity $u$. The gravitational acceleration is $g$ and the air exerts a drag force on the insect equal to $mk$ times the velocity of the wind relative to the insect.

(a) Show that the path of the insect is given by

$$x = \left(u + \frac{g}{k}\right)t + \frac{1 - e^{-kt}}{k} \left(v - u - \frac{g}{k}\right).$$
(b) In the case where the insect jumps vertically in a horizontal wind, show that the time $T$ that elapses before it returns to the ground (which is also horizontal) satisfies

$$1 - e^{-kT} = \frac{kT}{1 + \lambda},$$

where $\lambda = kv/g$. Find an expression for the horizontal range $R$ in terms of $\lambda$, $u$ and $T$. (Here $v = |v|$, $g = |g|$ and $u = |u|$.)

7. A ball of mass $m$ moves, under gravity, in a resistive medium that produces a frictional force of magnitude $kv^2$, where $v$ is the ball’s speed. If the ball is projected vertically upwards with initial speed $u$, show by dimensional analysis that when the ball returns to its point of projection, its speed $w$ can be written in the form

$$w = uf(\lambda),$$

where $\lambda = ku^2/mg$.

Integrate the equation of motion to show that $f(\lambda) = (1 + \lambda)^{-1/2}$. [Hint: Thinking about $v$ as a function of time may not be the easiest approach.] Discuss what happens in the two extremes $\lambda \gg 1$ and $\lambda \ll 1$.

8*. The temperature $\theta(x, t)$ in a very long rod is governed by the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2},$$

where $D$ is a constant (the thermal diffusivity of the rod). At time $t = 0$, the point $x = 0$ is heated to a high temperature. At all later times, the conservation of heat energy implies that

$$Q = \int_{-\infty}^{\infty} \theta(x, t) \, dx$$

is constant. Use dimensional analysis to show that $\theta(x, t)$ can be written in the form

$$\theta(x, t) = \frac{Q}{\sqrt{Dt}} F(z),$$

where $z = x/\sqrt{Dt}$, and show further that

$$\frac{d^2 F}{dz^2} + \frac{z}{2} \frac{dF}{dz} + \frac{1}{2} F = 0.$$ 

Integrate this equation once directly to obtain a first-order differential equation. Evaluate the constant of integration by considering either the symmetry of the problem or the behaviour of the solution as $z \to \pm \infty$. Hence show that, for $t > 0$,

$$\theta(x, t) = \frac{Q}{\sqrt{4\pi Dt}} \exp \left( -\frac{x^2}{4Dt} \right).$$

Please send any comments and corrections to gio10@cam.ac.uk