

## A7a Dynamics and Relativity: Examples Sheet 1

Lent 2023

Corrections and suggestions should be emailed to [S.A.Hartnoll@damtp.cam.ac.uk](mailto:S.A.Hartnoll@damtp.cam.ac.uk). Harder questions are starred and should be attempted after the others.

1. In an inertial frame, a particle moves along a trajectory  $\mathbf{x}(t)$  under the influence of a second heavy particle that is at rest at  $\mathbf{x}_o$ . The acceleration of the first particle is given by

$$m\ddot{\mathbf{x}}(t) = -k \frac{\mathbf{x}(t) - \mathbf{x}_o}{|\mathbf{x}(t) - \mathbf{x}_o|^d}. \quad (1)$$

Here  $m, k$  and  $d$  are constants. We may now consider this system in a different frame of reference. All positions in the new frame are related to positions in the old frame by  $\mathbf{x}' = f(\mathbf{x}, t)$ . Both frames use the same clock, so that  $t' = t$ . Consider, in particular, the transformations (a) – (c) listed below. In each case obtain an equation for  $m\ddot{\mathbf{x}}'(t')$ . Note that  $'$  denotes the new frame and not a derivative.

- (a)  $\mathbf{x}' = R\mathbf{x}$ , where the matrix  $R$  obeys  $R^T R = 1$
- (b)  $\mathbf{x}' = \mathbf{x} + \mathbf{b}$ , with  $\mathbf{b}$  a constant vector
- (c)  $\mathbf{x}' = \mathbf{x} - \mathbf{v}t$ , with  $\mathbf{v}$  a constant vector

How do your results relate to the Galilean invariance of physical laws?

Certain special forces may enjoy additional invariances. For what value of  $d$  is equation (1) invariant under the *Galilean spacetime rescaling*  $\mathbf{x}' = \lambda\mathbf{x}$  and  $t' = \lambda^2 t$ , with  $\lambda$  a constant?

In lectures we obtained the acceleration due to gravity on Earth as

$$\ddot{\mathbf{x}}(t) = \mathbf{g},$$

with  $\mathbf{g}$  a fixed vector pointing downwards. Is this equation invariant under Galilean transformations? If not, why does it not violate the principle of Galilean relativity?

2. A particle of mass  $m$  at position  $\mathbf{x}$  experiences a force

$$\mathbf{F} = \left( -\frac{a}{r^2} + \frac{2b}{r^3} \right) \hat{\mathbf{x}}$$

where  $\hat{\mathbf{x}}$  is the unit vector in the radial direction, the magnitude  $r = |\mathbf{x}|$ , and  $a$  and  $b$  are positive constants. Show, by finding a potential  $V(r)$  such that  $\mathbf{F} = -\nabla V$ , that  $\mathbf{F}$  is conservative. (Hint: you will need the result  $\nabla r = \hat{\mathbf{x}}$ ).

Sketch the potential  $V(r)$  and describe qualitatively the possible motions of the particle moving in the radial direction, considering different starting positions and speeds. If the particle starts at the point  $r = 2b/a$ , what is the minimum speed that the particle must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance  $R$ , much larger than the Earth's radius. Treating the Earth as a point of mass  $M$ , use dimensional analysis to show that the time  $T$  taken by the satellite to reach the Earth is given by

$$T = C \left( \frac{R^3}{GM} \right)^{\frac{1}{2}},$$

where  $G$  is the gravitational constant and  $C$  is a dimensionless constant. (The acceleration due to the Earth's gravitational field at a distance  $r$  from the centre of the Earth is  $GM/r^2$ ).

What is the equation of energy conservation for the satellite? By solving this differential equation, show that  $C = \pi/2\sqrt{2}$ .

4. A long time ago, in a galaxy far far away, a perfectly spherical Death Star was constructed. Its surrounding force field caused a particle at position  $\mathbf{x}$  relative to the Death Star to experience an acceleration

$$\ddot{\mathbf{x}} = \lambda \mathbf{x} \times \dot{\mathbf{x}}$$

where  $\lambda$  is a constant. Show that particles move in this field with constant speed. Show, moreover, that the magnitude of acceleration is also constant.

- (a) A particle is projected *radially* with speed  $v$  from a point  $\mathbf{x} = R\hat{\mathbf{x}}$  on the surface of the Death Star. Show that its trajectory is given by

$$\mathbf{x} = (vt + R)\hat{\mathbf{x}}$$

- (b) By considering the second derivative of  $\mathbf{x} \cdot \mathbf{x}$  show that, for any particle moving in the force field, the distance  $x$  to the centre of the Death Star is given by

$$x^2 = v^2 \left( (t - t_0)^2 + t_1^2 \right)$$

where  $t_0$  and  $t_1$  are constants and  $v$  is the speed of the particle. Obtain an expression for  $\mathbf{x} \cdot \dot{\mathbf{x}}$  and show that  $|\ddot{\mathbf{x}}| = \lambda t_1 v^2$ .

5. A particle of mass  $m$ , charge  $q$  and position  $\mathbf{x}$  moves in a constant, uniform field  $\mathbf{B}$  which points in a horizontal direction. The particle is also under the influence of gravity,  $\mathbf{g}$ , acting vertically downwards. Write down the equation of motion and show that it is invariant under translations  $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{x}_0$ . Obtain

$$\dot{\mathbf{x}} = \alpha \mathbf{x} \times \mathbf{n} + \mathbf{g}t + \mathbf{a}$$

where  $\alpha = qB/m$ ,  $\mathbf{n}$  is a unit vector in the direction of  $\mathbf{B}$  and  $\mathbf{a}$  is a constant vector. Show that, with a suitable choice of origin,  $\mathbf{a}$  can be written in the form  $\mathbf{a} = a\mathbf{n}$ .

By choosing suitable axes, show that the particle undergoes a helical motion with a constant horizontal drift.

Suppose that you now wish to eliminate the drift by imposing a uniform electric field  $\mathbf{E}$ . Determine the direction and magnitude of  $\mathbf{E}$ .

6. At time  $t = 0$ , an insect of mass  $m$  jumps from a point  $O$  on the ground with velocity  $\mathbf{v}$ , while a wind blows with velocity  $\mathbf{u}$ . The gravitational acceleration is  $\mathbf{g}$  and the air exerts a retarding force on the insect equal to  $mk$  times the velocity of the wind relative to the insect.

(a) Show that the path of the insect is given by

$$\mathbf{x} = (\mathbf{u} + \mathbf{g}/k)t + \frac{1 - e^{-kt}}{k} (\mathbf{v} - \mathbf{u} - \mathbf{g}/k)$$

(b) In the case where the insect jumps vertically in a horizontal wind, show that the time  $T$  that elapses before it returns to earth satisfies

$$(1 - e^{-kT}) = \frac{kT}{1 + \gamma}$$

where  $\gamma = kv/g$ . Find an expression for the range  $R$  in terms of  $\gamma$ ,  $u$  and  $T$ . (Here  $v = |\mathbf{v}|$ ,  $g = |\mathbf{g}|$ , and  $u = |\mathbf{u}|$ .)

7. A ball of mass  $m$  moves in a resisting medium that produces a friction force of magnitude  $kv^2$ , where  $v$  is the ball's speed. If the ball is projected vertically upwards with initial speed  $u$ , show by dimensional analysis that when the ball returns to its point of projection, its speed  $w$  can be written in the form

$$w = uf(\lambda),$$

where  $\lambda = ku^2/mg$ . Integrate the equations of motion to show that  $f(\lambda) = (1 + \lambda)^{-1/2}$ . Discuss what happens in the two extremes  $\lambda \gg 1$ , and  $\lambda \ll 1$ .

[Hint: Thinking about  $v$  as a function of time may not be the easiest approach]

8. (\*) The temperature  $\theta(x, t)$  in a very long rod is governed by the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2}$$

where  $D$  is a constant (the *thermal diffusivity* of the rod). At time  $t = 0$ , the point  $x = 0$  is heated to a high temperature. At all later times, the conservation of heat energy implies that

$$Q = \int_{-\infty}^{\infty} \theta(x, t) dx$$

is constant. Use dimensional analysis to show that  $\theta(x, t)$  can be written in the form

$$\theta(x, t) = \frac{Q}{\sqrt{Dt}} F(z)$$

where  $z = x/\sqrt{Dt}$  and show further that

$$\frac{d^2 F}{dz^2} + \frac{z}{2} \frac{dF}{dz} + \frac{1}{2} F = 0$$

Integrate this equation once directly to obtain a first order differential equation (you may assume that  $zF(z) \rightarrow 0$  and  $dF(z)/dz \rightarrow 0$  as  $z \rightarrow \infty$ ), and hence show that

$$\theta(x, t) = \frac{Q}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}.$$