

### Example Sheet 1

[Example Sheet 1 is based on (approximately) Lectures 1-7 of the course.]

1. In one spatial dimension, two frames of reference  $S$  and  $S'$  have coordinates  $(x, t)$  and  $(x', t')$  respectively. The coordinates are related by

$$x' = f(x, t) \quad \text{and} \quad t' = t.$$

Viewed in frame  $S$ , a particle follows a trajectory  $x = x(t)$ . It has velocity  $v = dx/dt$  and acceleration  $a = d^2x/dt^2$ . Viewed in  $S'$ , the trajectory is  $x' = f(x(t), t)$ . Using the chain rule, show that the velocity and acceleration of the particle in  $S'$  are given by

$$v' = \frac{dx'}{dt'} = v \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}, \quad a' = \frac{d^2x'}{dt'^2} = a \frac{\partial f}{\partial x} + v^2 \frac{\partial^2 f}{\partial x^2} + 2v \frac{\partial^2 f}{\partial x \partial t} + \frac{\partial^2 f}{\partial t^2}.$$

Suppose now that both  $S$  and  $S'$  are inertial frames. Explain why the function  $f$  must obey  $\partial^2 f / \partial x^2 = \partial^2 f / \partial x \partial t = \partial^2 f / \partial t^2 = 0$ . What is the most general form of  $f$  with these properties? Interpret this result.

2. A particle of mass  $m$  experiences a force field

$$\mathbf{F}(\mathbf{r}) = \left( -\frac{a}{r^2} + \frac{2b}{r^3} \right) \hat{\mathbf{r}},$$

where  $\hat{\mathbf{r}} = \mathbf{r}/r$  is a unit vector in the radial direction and  $a$  and  $b$  are positive constants. Show, by finding a potential energy  $V(r)$  such that  $\mathbf{F} = -\nabla V$ , that  $\mathbf{F}$  is conservative. (You will need the result  $\nabla r = \hat{\mathbf{r}}$ .)

Sketch  $V(r)$  and describe qualitatively the possible motions of the particle moving in the radial direction, considering different initial positions and velocities. If the particle starts at  $r = 2b/a$ , what is the minimum speed that it must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance  $R$ , much larger than the Earth's radius. Treating the Earth as a point of mass  $M$ , use dimensional analysis to show that the time  $T$  taken by the satellite to reach the Earth is given by

$$T = C \left( \frac{R^3}{GM} \right)^{1/2},$$

where  $G$  is the gravitational constant and  $C$  is a dimensionless constant. (You will need the fact that the acceleration due to the Earth's gravitational field at a distance  $r$  from the centre of the Earth is  $GM/r^2$ .)

What is the equation of energy conservation for the satellite? By solving this differential equation, show that  $C = \pi/2\sqrt{2}$ .

4. A long time ago in a galaxy far, far away, a Death Star was constructed. Its surrounding force field caused any particle at position  $\mathbf{r}$  relative to the centre of the Death Star to experience an acceleration

$$\ddot{\mathbf{r}} = \lambda \mathbf{r} \times \dot{\mathbf{r}},$$

where  $\lambda$  is a constant. Show that the particle moves in this force field with constant speed. Show also that the magnitude of its acceleration is constant.

(a) A particle is projected *radially* with speed  $v$  from a point  $\mathbf{r} = R \hat{\mathbf{r}}$  on the surface of the Death Star. Show that its trajectory is given by

$$\mathbf{r} = (R + vt) \hat{\mathbf{r}}.$$

(b) By considering the second derivative of  $\mathbf{r} \cdot \mathbf{r}$  show that, for any particle moving in the force field, the distance  $r$  from the centre of the Death Star is given by

$$r^2 = v^2(t - t_0)^2 + r_0^2,$$

where  $t_0$  and  $r_0$  are constants and  $v$  is the speed of the particle. Obtain an expression for  $\mathbf{r} \cdot \dot{\mathbf{r}}$  and show that  $|\ddot{\mathbf{r}}| = \lambda r_0 v$ .

5. A particle of mass  $m$ , charge  $q$  and position  $\mathbf{x}(t)$  moves in both a uniform magnetic field  $\mathbf{B}$ , which points in a horizontal direction, and a uniform gravitational field  $\mathbf{g}$ , which points vertically downwards. Write down the equation of motion and show that it is invariant under translations  $\mathbf{x} \mapsto \mathbf{x} + \mathbf{x}_0$ . Show that

$$\dot{\mathbf{x}} = \omega \mathbf{x} \times \mathbf{n} + \mathbf{g}t + \mathbf{a},$$

where  $\omega = qB/m$  is the gyrofrequency,  $\mathbf{n}$  is a unit vector in the direction of  $\mathbf{B}$ , and  $\mathbf{a}$  is a constant vector. Show also that, with a suitable choice of origin,  $\mathbf{a}$  can be written in the form  $\mathbf{a} = a \mathbf{n}$ .

By choosing suitable axes, show that the particle undergoes a helical motion together with a constant horizontal drift perpendicular to  $\mathbf{B}$ .

Suppose that you now wish to eliminate the drift by imposing a uniform electric field  $\mathbf{E}$ . Determine the direction and magnitude of  $\mathbf{E}$ .

6. At time  $t = 0$ , an insect of mass  $m$  jumps from a point  $O$  on the ground with velocity  $\mathbf{v}$ , while a wind blows with constant velocity  $\mathbf{u}$ . The gravitational acceleration is  $\mathbf{g}$  and the air exerts a drag force on the insect equal to  $mk$  times the velocity of the wind *relative to the insect*.

(a) Show that the path of the insect is given by

$$\mathbf{x} = \left( \mathbf{u} + \frac{\mathbf{g}}{k} \right) t + \frac{1 - e^{-kt}}{k} \left( \mathbf{v} - \mathbf{u} - \frac{\mathbf{g}}{k} \right).$$

(b) In the case where the insect jumps vertically in a horizontal wind, show that the time  $T$  that elapses before it returns to the ground (which is also horizontal) satisfies

$$1 - e^{-kT} = \frac{kT}{1 + \lambda},$$

where  $\lambda = kv/g$ . Find an expression for the horizontal range  $R$  in terms of  $\lambda$ ,  $u$  and  $T$ . (Here  $v = |\mathbf{v}|$ ,  $g = |\mathbf{g}|$  and  $u = |\mathbf{u}|$ .)

7. A ball of mass  $m$  moves, under gravity, in a resistive medium that produces a frictional force of magnitude  $kv^2$ , where  $v$  is the ball's speed. If the ball is projected vertically upwards with initial speed  $u$ , show by dimensional analysis that when the ball returns to its point of projection, its speed  $w$  can be written in the form

$$w = uf(\lambda),$$

where  $\lambda = ku^2/mg$ .

Integrate the equation of motion to show that  $f(\lambda) = (1 + \lambda)^{-1/2}$ . [Hint: Thinking about  $v$  as a function of time may not be the easiest approach.]. Discuss what happens in the two extremes  $\lambda \gg 1$  and  $\lambda \ll 1$ .

8\*. The temperature  $\theta(x, t)$  in a very long rod is governed by the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2},$$

where  $D$  is a constant (the *thermal diffusivity* of the rod). At time  $t = 0$ , the point  $x = 0$  is heated to a high temperature. At all later times, the conservation of heat energy implies that

$$Q = \int_{-\infty}^{\infty} \theta(x, t) dx$$

is constant. Use dimensional analysis to show that  $\theta(x, t)$  can be written in the form

$$\theta(x, t) = \frac{Q}{\sqrt{Dt}} F(z),$$

where  $z = x/\sqrt{Dt}$ , and show further that

$$\frac{d^2 F}{dz^2} + \frac{z}{2} \frac{dF}{dz} + \frac{1}{2} F = 0.$$

Integrate this equation once directly to obtain a first-order differential equation. Evaluate the constant of integration by considering either the symmetry of the problem or the behaviour of the solution as  $z \rightarrow \pm\infty$ . Hence show that, for  $t > 0$ ,

$$\theta(x, t) = \frac{Q}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right).$$

Please send any comments and corrections to [pjh1@cam.ac.uk](mailto:pjh1@cam.ac.uk)