Dynamics and Relativity: Example Sheet 2

Professor David Tong, February 2012

1. In a system of particles, the ith particle has mass m_i and position vector $\mathbf{x}_i$ with respect to a fixed origin. The centre of mass of the system is at $\mathbf{R}$. Show that $\mathbf{L}$, the total angular momentum of the system about the origin, and $\mathbf{L}_{\text{CoM}}$, the total angular momentum of the system about the centre of mass, are related by

$$\mathbf{L}_{CoM} = \mathbf{L} - \mathbf{R} \times \mathbf{P}$$

where $\mathbf{P}$ is the total linear momentum of the system.

Given that $d\mathbf{P}/dt = \mathbf{F}$ where $\mathbf{F}$ is the total external force and $d\mathbf{L}/dt = \boldsymbol{\tau}$ where $\boldsymbol{\tau}$ is the total external torque about the origin, show that

$$\frac{d\mathbf{L}_{CoM}}{dt} = \boldsymbol{\tau}_{CoM},$$

where $\boldsymbol{\tau}_{CoM}$ is the total external torque about the centre of mass.

2. A system of particles with masses m_i and position vectors $\mathbf{x}_i$, $i = 1, \ldots, n$, moves under its own mutual gravitational attraction alone. Write down the equation of motion for $\mathbf{x}_i$. Show that a possible solution of the equations of motion is given by $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are constant vectors satisfying

$$\mathbf{a}_i = \frac{9G}{2} \sum_{j \neq i} \frac{m_j(\mathbf{a}_i - \mathbf{a}_j)}{|\mathbf{a}_i - \mathbf{a}_j|^3}.$$

Show that, for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other fixed point?

3. A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive the equation of motion

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - gm$$

where v and m are the speed and total mass of the rocket (including fuel) at time t. If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0 e^{-(gt+v)/u}.$$

4. A firework of initial mass m_0 is fired vertically upwards from the ground. The rate of burning of fuel $dm/dt = -\alpha$ and the fuel is ejected at constant speed u relative to the firework. Show that the speed of the firework at time t, where $0 < t < m_0/\alpha$, is

$$v(t) = -gt - u \log\left(1 - \frac{\alpha t}{m_0}\right)$$

and that this is positive provided $u > m_0 g/\alpha$.

Suppose now that nearly all of the firework consists of fuel, the mass of the containing shell being negligible. Show that the height attained by the shell when all of the fuel is burnt is

$$\frac{m_0}{\alpha}\left(u - \frac{m_0 g}{2\alpha}\right)$$

5. Two particles of masses m_1 and m_2 move under their mutual gravitational attraction. Show from first principles that the quantity

$$\frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{GM}{r}$$

is constant, where $\mathbf{r}$ is the position vector of one particle relative to the other and $M = m_1 + m_2$.

The particles are released from rest a long way apart, and fall towards each other. Show that the position of their centre of mass is fixed, and that when they are a distance r apart their relative speed is $\sqrt{2GM/r}$.

When the particles are a distance a apart, they are given equal and opposite impulses (change of momentum) each of magnitude I, and each perpendicular to the direction of motion. Show that subsequently $r^2\omega = aI/\mu$, where ω is the angular speed of either particle relative to the centre of mass and μ is the reduced mass of the system.

Show further that the minimum separation, d, of the two particles in the subsequent motion satisfies

$$(a^2 - d^2)I^2 = 2GM\mu^2 d$$

6. A particle moves in a fixed plane and its position vector at time t is $\mathbf{r}$. Let (r, θ) be plane polar coordinates and let $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the direction of increasing r and increasing θ respectively. Show that

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

The particle moves outwards with speed v on the equi-angular spiral $r = a\exp(\theta\cot\alpha)$, where a and α are constants, with $0 < \alpha < \frac{1}{2}\pi$. Show that

$$v\sin\alpha = r\dot{\theta}$$

and hence that

$$\dot{\mathbf{r}} = v\cos\alpha\,\hat{\mathbf{r}} + v\sin\alpha\,\hat{\boldsymbol{\theta}}$$

Give an expression for $\ddot{\mathbf{r}}$ and show that $|\ddot{\mathbf{r}}|^2 = \dot{v}^2 + v^2\dot{\theta}^2$.

(*) If $\dot{\theta}$ takes a constant value ω, show that the acceleration has magnitude v^2/r and is directed at an angle 2α to the position vector.

7. For a particle subject to an inverse square force $\mathbf{F} = -mk\hat{\mathbf{r}}/r^2$, the vectors $\mathbf{h}$ and $\mathbf{e}$ are defined by

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{k} - \frac{\mathbf{r}}{r}$$

Show that $\mathbf{h}$ is constant and deduce that the particle moves in a plane through the origin.

The vector $\mathbf{e}$ is known as the *Laplace-Runge-Lenz* vector. Show that it too is is constant and that

$$er\cos\theta = \frac{h^2}{k} - r$$

where $e = |\mathbf{e}|$, $h = |\mathbf{h}|$ and θ is the angle between $\mathbf{r}$ and $\mathbf{e}$. Deduce that the orbit is a conic section.

8. A particle of unit mass moves with speed v in the gravitational field of the Sun and is influenced by radiation pressure. The forces acting on the particle are μ/r^2 towards the sun and kv opposing the motion, where μ and k are constants. Write down the vector equation of motion and show that the vector $\mathbf{H}$, defined by

$$\mathbf{H} = e^{kt}\,\mathbf{r} \times \dot{\mathbf{r}}$$

is constant. Deduce that the particle moves in a plane through the origin.

Establish the equations

$$r^2\dot{\theta} = h\mathrm{e}^{-kt} \qquad \text{and} \qquad \mu r = h^2\mathrm{e}^{-2kt} - r^3(\ddot{r} + k\dot{r})$$

where r and θ are plane polar coordinates centred on the Sun and h is a constant.

Show that, when $k = 0$, a circular orbit of radius a exists for any value of a, and find its angular frequency ω in terms of a and μ.

When $k/\omega \ll 1$, r varies so slowly that $\dot{r}$ and $\ddot{r}$ may be neglected in the above equations. Verify that in this case an approximate solution is

$$r = a\mathrm{e}^{-2kt}, \qquad \dot{\theta} = \omega\mathrm{e}^{3kt}$$

Give a brief qualitative description of the behaviour of this solution for $t > 0$. Does the speed of the particle increase or decrease?

9*. A particle P of mass m moves under the influence of a central force of magnitude mk/r^3 directed towards a fixed point O. Initially $r = a$ and P has velocity v perpendicular to OP, where $v^2 < k/a^2$. Prove that P spirals in towards O (you should give the geometric equation of the spiral). Show also that it reaches O in a time

$$T = \frac{a^2}{\sqrt{k - a^2v^2}}$$