A7c Dynamics and Relativity: Examples Sheet 3 Lent 2025

Corrections and suggestions should be emailed to `S.A.Hartnoll@damtp.cam.ac.uk`. *Harder questions are starred and should be attempted after the others.*

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1. In a system of particles, the ith particle has mass m_i and position vector $\mathbf{x}_i$ with respect to a fixed origin. The centre of mass of the system is at $\mathbf{R}$. Show that $\mathbf{L}$, the total angular momentum of the system about the origin, and $\mathbf{L}_{\mathrm{CoM}}$, the total angular momentum of the system about the centre of mass, are related by

$$\mathbf{L}_{CoM} = \mathbf{L} - \mathbf{R} \times \mathbf{P}$$

where $\mathbf{P}$ is the total linear momentum of the system.

Given that $d\mathbf{P}/dt = \mathbf{F}$ where $\mathbf{F}$ is the total external force and $d\mathbf{L}/dt = \boldsymbol{\tau}$ where $\boldsymbol{\tau}$ is the total external torque about the origin, show that

$$\frac{d\mathbf{L}_{CoM}}{dt} = \boldsymbol{\tau}_{CoM},$$

where $\boldsymbol{\tau}_{CoM}$ is the total external torque about the centre of mass.

2. A system of particles with masses m_i and position vectors $\mathbf{x}_i$, $i = 1, \ldots, n$, moves under its own mutual gravitational attraction alone. Write down the equation of motion for $\mathbf{x}_i$. Show that a possible solution of the equations of motion is given by $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are constant vectors satisfying

$$\mathbf{a}_i = \frac{9G}{2} \sum_{j \neq i} \frac{m_j(\mathbf{a}_i - \mathbf{a}_j)}{|\mathbf{a}_i - \mathbf{a}_j|^3}.$$

Show that, for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other fixed point?

(∗) What physical system might this setup describe?

3. A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive the equation of motion

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - gm$$

where v and m are the speed and total mass of the rocket (including fuel) at time t. If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0 e^{-(gt+v)/u}.$$

4. A firework of initial mass m_0 is fired vertically upwards from the ground. The rate of burning of fuel $dm/dt = -\alpha$ and the fuel is ejected at constant speed u relative to the firework. Show that the speed of the firework at time t, where $0 < t < m_0/\alpha$, is

$$v(t) = -gt - u \log\left(1 - \frac{\alpha t}{m_0}\right)$$

and that this is positive provided $u > m_0 g/\alpha$.

Suppose now that nearly all of the firework consists of fuel, the mass of the containing shell being negligible. Show that the height attained by the shell when all of the fuel is burnt is

$$\frac{m_0}{\alpha}\left(u - \frac{m_0 g}{2\alpha}\right)$$

5. (a) Thin circular discs of radius a and b are made of uniform materials with mass per unit area ρ_a and ρ_b, respectively. They lie in the same plane. Their centres A and B are connected by a light rigid rod of length c. Find the moment of inertia of the system about an axis through B perpendicular to the plane of the discs.

   (b) A thin uniform circular disc of radius a and centre A has a circular hole cut in it of radius b and centre B, where $AB = c < a - b$. The disc is free to oscillate in a vertical plane about a smooth fixed horizontal circular rod of radius b passing through the hole. Using the result of part (i), with ρ_b suitably chosen, show that the period of small oscillations is $2\pi\sqrt{l/g}$, where $l = c + (a^4 - b^4)/(2a^2c)$.

6. A yo-yo consists of two uniform discs, each of mass M and radius R, connected by a short light axle of radius a around which a portion of a thin string is wound. One end of the string is attached to the axle and the other to a fixed point P. The yo-yo is held with its centre of mass vertically below P and then released.

   Assuming that the unwound part of the string remains approximately vertical, use the principle of conservation of energy to find the equation of motion of the centre of mass of the yo-yo. Find the tension in the string as the yo-yo falls.

   If L is the length of string originally wound around the yo-yo, what is the speed of the yo-yo just before it reaches the end? Explain what happens next. What is the impulse (i.e. the time-integrated force) due to the tension in the string at this time?

7. A uniform circular cylinder of mass M and radius a is free to turn about its axis which is horizontal. A thin uniform cylindrical shell of mass $M/2$ and radius a is fitted over the cylinder. At time $t = 0$ the angular velocity of the cylinder is Ω, while the shell is at rest. The shell exerts a frictional torque on the cylinder of magnitude $k(\omega - \bar{\omega})$, where $\omega(t)$ and $\bar{\omega}(t)$ are the angular velocities of the cylinder and shell, respectively, at time t about the axis. Prove that

$$\omega(t) = \frac{1}{2}\Omega\left[1 + \exp\left(\frac{-4kt}{Ma^2}\right)\right],$$

   and find the corresponding expression for $\bar{\omega}(t)$.

8. A bullet of mass m is fired from a point $\mathbf{r}_0$ with velocity $\mathbf{u}$ in a frame which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame. The bullet is subject to a gravitational force $m\mathbf{g}$ which is constant in the rotating frame. Using the vector equation of motion and neglecting terms of order $|\boldsymbol{\omega}|^2$, show that the bullet's position vector measured in the rotating frame is approximately

$$\mathbf{r}_0 + \mathbf{u}t + (\frac{1}{2}\mathbf{g} - \boldsymbol{\omega} \times \mathbf{u})t^2 + \frac{1}{3}\mathbf{g} \times \boldsymbol{\omega}t^3$$

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at time t. Suppose that the bullet is projected from sea level on the Earth at latitude θ in the Northern hemisphere, at an angle $\pi/4$ from the upward vertical and in a Northward direction. Show that when the particle returns to sea level (neglecting the curvature of the Earth's surface), it has been deflected to the East by an amount approximately equal to

$$\frac{\sqrt{2}\,\omega|\mathbf{u}|^3}{3g^2}(3\sin\theta - \cos\theta)$$

where ω is the angular speed of the Earth. Evaluate the approximate size of this deflection at latitude $52°$ N for $|\mathbf{u}| = 1000\,\mathrm{m/s}$.

9. A square hoop $ABCD$ is made of fine smooth wire and has side length $2a$. The hoop is horizontal and rotating with constant angular speed ω about a vertical axis through A. A small bead which can slide on the wire is initially at rest, relative to the hoop, at the midpoint of the side BC. Choose axes fixed relative to the hoop, and let y be the distance of the bead from the vertex B on the side BC. Write down the position vector of the bead in the rotating frame.

   Using the standard expression for acceleration in a rotating frame, show that

$$\ddot{y} - \omega^2 y = 0$$

   Hence show that the time which the bead takes to reach a corner of the hoop is $\omega^{-1}\cosh^{-1} 2$. Using dimensional analysis, explain why this time is independent of a.

   Obtain an expression for the magnitude of the force exerted by the hoop on the bead.

10. In this sequence of questions on the Coriolis force, use ω for the angular speed of the Earth, assume that events take place at latitude θ in the northern hemisphere and ignore centrifugal forces.

    (a) Are bath-plug vortices in the northern hemisphere likely, on average, to be clockwise or anticlockwise?

    (b) A straight river flows with speed v in a direction α degrees East of North. Show that the effect of the coriolis force is to undermine the right bank. Does the magnitude of the effect depend on α?

    (c) A plumb line is attached to the ceiling inside one of the carriages of a train and hangs down freely, at rest relative to the train. When the train is travelling at speed V in the north-easterly direction the plumb line hangs at an angle ϕ to the direction in which it hangs when the train is at rest. Ignoring centrifugal forces, show that $\phi \approx (2\omega V \sin\theta)/g$. Why can the centrifugal force be ignored?

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