Mathematical Tripos 2004, Part IB : Electromagnetism, Lent. 2004
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1 Introduction

1.1 Electric Charge

The existence of electric charge was well-known already to the ancient Greeks, from the rubbing of amber with fur.

Experiments show that there are charges of two kinds, positive and negative. All stable charged matter owes its charge to a preponderance of electrons, if negative, and of protons, if positive. In fact, each electron and each proton carry a charge $\mp e$, where

$$e = 1.6 \times 10^{-19}\, C, \quad (C = \text{Coulomb}), \tag{1}$$

a magnitude so small that total charge can be regarded as a continuous variable. Thus we can refer to the charge density $\rho(\mathbf{r})$ as the charge per unit volume at a point $\mathbf{r}$ of a spatial distribution of charge.

Experiment shows also that, when we consider stationary particles P_1 and P_2 situated at $\mathbf{r}_1$ and $\mathbf{r}_2$ with charges q_1 and q_2, then P_1 experiences a force

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1\, q_2}{{r_{12}}^2}\,\frac{\mathbf{r}_{12}}{r_{12}} = \frac{1}{4\pi\epsilon_0}\frac{q_1\, q_2}{{r_{12}}^2}\hat{\mathbf{r}}_{12}, \tag{2}$$

due to P_2. This expresses the inverse-square or Coulomb law. Here

$$\mathbf{r}_{12} = -\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2, \quad r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \quad \hat{\mathbf{r}}_{12} = \mathbf{r}_{12}/r_{12}, \tag{3}$$

with $\hat{\mathbf{r}}_{12}$ a unit vector pointing from P_2 to P_1.

If $q_1\, q_2$ is positive (same sign charges) then $\mathbf{F}_{12}$ is an repulsive force; if negative (opposite sign charges), then it is attractive.

The factor $\dfrac{1}{4\pi\epsilon_0}$ is a dimensional quantity arising because of our use of SI or Système Internationale units (=MKS, metre, kilogram, second units).

Next we consider the force on charge q_1 at $\mathbf{r}_1$ due to a set of charges q_j at $\mathbf{r}_j$. This is given by

$$\mathbf{F}_1 = \frac{q_1}{4\pi\epsilon_0}\sum_{j\neq 1}\frac{q_j \mathbf{r}_{1j}}{{r_{1j}}^3}. \tag{4}$$

Hence, for the force on a charge q at $\mathbf{r}$ due to charge of density $\rho(\mathbf{r}')$ continuously distributed over a spatial volume V, we have

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3} \quad . \tag{5}$$

Now we define the electric field $\mathbf{E}(\mathbf{r})$ of such a distribution of charge to be the force it would exert on a unit charge if one were to be placed at $\mathbf{r}$, *i.e.*

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{6}$$

Similarly for a system of point charge q_j at $\mathbf{r}_j$, and to charge of density $\sigma(\mathbf{r}')$ distributed over a surface S, we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j(\mathbf{r}-\mathbf{r}_j)}{|\mathbf{r}-\mathbf{r}_j|^3} \tag{7}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{(\mathbf{r}-\mathbf{r}')\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} dS'. \tag{8}$$

Ex. q at the origin O gives rise to the electric field $\mathbf{E}(\mathbf{r})$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r},\ r = |\mathbf{r}|,\ |\hat{\mathbf{r}}| = 1. \tag{9}$$

and q', if placed at $\mathbf{r}$, would experience a force (due to this field),

$$\mathbf{F}(\mathbf{r}) = q'\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\,q'}{r^2}\hat{\mathbf{r}}. \tag{10}$$

1.2 Electric current

The ancient Greeks were well-aware too of magnetic material like lodestone, and of its effects. However a modern view is that the magnetic field $\mathbf{B}(\mathbf{r})$ and related forces are due to charges in motion, *i.e.* to electric currents. So we look next at the idea of electric current.

There are very many types of electric current flow. Here we confine ourselves to getting an intuitive picture of current flow in a copper wire.

First we recall that an atom is an electrically neutral system with a central nuclei containing Z protons with Z electrons moving around it 'in orbits' governed by the laws of quantum mechanics.

We use a battery to apply an electric field to a length of copper wire (or similarly to some suitable piece of crystalline material capable of conducting an electric current). Then some of the electrons of the copper atoms of the wire are detached from the atoms, leaving them as positively charged ions. These ions are held in position by the mechanical forces that describe the constitution of the material, and the detached electrons are moved like a gas, by the applied electric field, through the essentially fixed ionic background. In other words the detached electrons (called conduction electrons) constitute an electric current flowing in the wire (material).

Suppose we have a distribution of charge carriers, here electrons of charge q, N per unit volume, whose average motion is a drift velocity $\mathbf{v}$.

This distribution has charge density $\rho = Nq$, and constitutes electric current flow of current density $\mathbf{J} = Nq\mathbf{v} = \rho\mathbf{v}$. To see that (or *how*) $\mathbf{J}$ describes the rate of flow of electric charge, let $\delta\mathbf{S}$ be a small plane element of area, and let $\mathcal{C}$ be a cylinder of current flow of cross-section $\delta\mathbf{S}$ with generators parallel to $\mathbf{v}$ of magnitude $|\mathbf{v}|$. Then $\mathcal{C}$ has volume $\mathbf{v}.\delta\mathbf{S}$, contains charge $Nq\mathbf{v}.\delta\mathbf{S} = \mathbf{J}.\delta\mathbf{S}$, and all of this charge flows across $\delta\mathbf{S}$ in unit time. Thus, writing $\delta\mathbf{S} = \delta S\mathbf{n}$, we see that in the current flow of current density $\mathbf{J}$, an amount of charge $\mathbf{J}.\mathbf{n}$ crosses unit area perpendicular to $\mathbf{n}$ in unit time.

The total charge per unit time passing through a surface S is called the electric current I through S

$$I = \int_S \mathbf{J}\cdot\mathbf{n}dS = \int_S \mathbf{J}\cdot\mathbf{dS}, \quad \mathbf{dS} = \mathbf{n}dS. \tag{11}$$

We comment here on the generic term *flux*: The flux f of a vector field $\mathbf{v}$ through a surface S is defined by

$$f = \int_S \mathbf{v} \cdot \mathbf{dS}. \tag{12}$$

Here S can either be closed bounding a spatial volume V, so that f is the flux of $\mathbf{v}$ out of $S = \partial V$, as in the Gauss theorem context of sec. 1.5 below, or else open and bounded by a curve $C = \partial S$, as in the definition just given, (11), of current I as the flux of current density through S, or through C. Physically what we have seen is that I measures the rate of flow of charge through S.

1.3 Magnetism

Magnetic fields $\mathbf{B}(\mathbf{r})$ arise from bar magnets, or from electric currents in wires, coils, etc. If a particle of charge q has position vector $\mathbf{r}$ and velocity $\mathbf{v} = \dot{\mathbf{r}}$, and moves in the presence of electric and magnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, it is an experimental fact that it experiences a force (the Lorentz force)

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}), \tag{13}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{r})$ and $\mathbf{B} = \mathbf{B}(\mathbf{r})$.

Consider the effect of the field $\mathbf{B}$ of the bar magnet on the wire. The current in the wire involves particles of charge q moving along the wire with velocity $\mathbf{v}$. Each one feels a (magnetic) force $q\mathbf{v}\wedge\mathbf{B}$ which, for positive qv, tends to push them downwards. One can see such a wire move downwards in experiment.

In a related experiment, one employs a fixed current carrying circuit connected to a battery instead of the bar magnet. In this case the second circuit experiences a force of attraction towards the first one due to the magnetic field of the first circuit. See Sec. 3.7.

One can also give the (magnetic) force per unit volume on a medium carrying N charges q per unit volume each moving with velocity $\mathbf{v}$

$$\mathbf{f} = Nq\mathbf{v}\wedge\mathbf{B} = \mathbf{J}\wedge\mathbf{B}. \tag{14}$$

1.4 Maxwell's Equations

It was the great achievement of Maxwell to unify the separate subjects electricity and magnetism into a single consistent formalism involving a set of equations (Maxwell's equations) capable of describing all classical electromagnetic phenomena. For charges and currents in a non-polarisable and non-magnetisable medium, such as the vacuum, these are

$$\begin{aligned}
\nabla\wedge\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 && (15)\\
\nabla\cdot\mathbf{B} &= 0 && (16)\\
\nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho && (17)\\
\nabla\wedge\mathbf{B} &= \mu_0(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}) && (18)
\end{aligned}$$

where ρ and $\mathbf{J}$ are the charge and current densities.

These equations involve two constants ϵ_0 and μ_0 that are not themselves of much physical significance (but see (46) below). The last term of (18) features the displacement current postulated by Maxwell in order to achieve a formalism that *consistently* unified previous theories of electricity and magnetism.

Certain more general media can be described by means of a suitable generalisation of the set (15–18) of Maxwell's equations, but this lies beyond the present course syllabus.

First we observe the consistency of Maxwell's equations. Since $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{F}) = 0$ for all vector fields $\mathbf{F}$, (15-16) imply

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{B}) = \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} \wedge \mathbf{E}) = 0. \tag{19}$$

So $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ is preserved in time.

Similarly $\boldsymbol{\nabla} \cdot (...)$ of (18) implies

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot \mathbf{J} + \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{E}) \\ 0 &= \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}. \end{aligned} \tag{20}$$

Here (17) has been used. Eq. (20) expresses the conservation of charge. Integrating (20) over a fixed volume V containing total charge Q

$$Q = \int_V \rho d\tau, \tag{21}$$

we derive

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t} d\tau = -\int_V \boldsymbol{\nabla} \cdot \mathbf{J} d\tau = -\int_{\partial V} \mathbf{J} \cdot \mathbf{dS}, \tag{22}$$

which states that the rate of decrease of the charge contained in V is equal to the flux of $\mathbf{J}$ out of V (through the surface $S = \partial V$). It is noted that the presence of the displacement term in (18) is essential in this demonstration of consistency.

1.5 Integral forms of Maxwell's equations

Maxwell's equations involve divs and curls. We can therefore convert them into useful integral forms by integrating over fixed volumes using the divergence theorem, or over fixed surfaces using Stokes's theorem.

$$\frac{\rho}{\epsilon_0} = \boldsymbol{\nabla} \cdot \mathbf{E} \Rightarrow \frac{1}{\epsilon_0} \int_V \rho d\tau = \int_V \boldsymbol{\nabla} \cdot \mathbf{E} d\tau \tag{23}$$

Hence

$$\frac{1}{\epsilon_0} Q = \int_{S=\partial V} \mathbf{E} \cdot \mathbf{dS}. \tag{24}$$

The right-hand side is the flux of $\mathbf{E}$ out of V. The statement (24) is Gauss's Law. It is of practical use.

Ex. Consider a point charge q at rest at O, and let V be the sphere of radius r centred at O. By symmetry the electric field must be of the form

$$\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r = E(r)\mathbf{n}, \tag{25}$$

so that

$$\int_{\partial V} \mathbf{E} \cdot \mathbf{dS} = \int_{\partial V} \mathbf{E} \cdot \mathbf{n} dS = E(r) \int_{\partial V} dS, \tag{26}$$

and hence

$$\begin{aligned} \frac{1}{\epsilon_0} q &= E(r)\, 4\pi r^2 \\ \mathbf{E} &= \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \mathbf{e}_r. \end{aligned} \tag{27}$$

Similarly (16) implies that

$$\int_{\partial V} \mathbf{B} \cdot \mathbf{dS} = 0, \tag{28}$$

for any closed surface $S = \partial V$. This can be interpreted as the statement that there are no magnetic 'charges' or magnetic monopoles.

Next (17) yields

$$\int_S \boldsymbol{\nabla} \wedge \mathbf{B} \cdot \mathbf{dS} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} + \mu_0 \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS}. \tag{29}$$

Hence, in the case of steady current (no time dependence), Stokes's theorem implies

$$\begin{aligned} \int_C \mathbf{B} \cdot \mathbf{dr} &= \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} \\ &= \mu_0 \text{ (flux of } \mathbf{J} \text{ through open } S \text{ bounded by } C) \\ &= \mu_0 I, \end{aligned} \tag{30}$$

where $I = \int_S \mathbf{J} \cdot \mathbf{dS}$ is the total current through S (or C). This is Ampère's Law. It too is useful in practice.

Ex. Consider an infinite straight wire lying along the z-axis and carrying a current I in the positive direction.

By symmetry, expect $\mathbf{B}$ of the form $\mathbf{B} = B(s)\mathbf{e}_\phi$ using cylindrical polars $(s,\ \phi,\ z)$. Then apply Ampère for C any circle centred on the z-axis and lying in a horizontal plane. On C we have

$$\mathbf{r} = s\mathbf{e}_s(\phi) \quad \text{so that, at constant } s, \quad \mathbf{dr} = s\mathbf{de}_s = s\frac{\partial \mathbf{e}_s}{\partial \phi} d\phi = s\mathbf{e}_\phi d\phi. \tag{31}$$

Then Ampère's law implies

$$B(s)s \int_0^{2\pi} d\phi = \mu_0 I \tag{32}$$

and hence

$$B(s) = \frac{\mu_0 I}{2\pi s}. \tag{33}$$

Finally (15) implies

$$\int_C \mathbf{E} \cdot \mathbf{dr} = \int_S \boldsymbol{\nabla} \wedge \mathbf{E} \cdot \mathbf{dS} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{34}$$

by applying Stokes's theorem to a fixed curve $C = \partial S$ bounding a fixed open surface S. If we define the electromotive force (or electromotance) acting in C by

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr}, \tag{35}$$

and the flux of $\mathbf{B}$ through (the open surface) S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{36}$$

then we get Faraday's Law of induction

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{37}$$

This will be studied later.

1.6 Electromagnetic waves

Here we consider Maxwell's equations in the absence of charges and of currents, *e.g.* in the vacuum.

$$\begin{aligned} \boldsymbol{\nabla}\wedge\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 & (38)\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 & (39)\\ \boldsymbol{\nabla}\cdot\mathbf{E} &= 0 & (40)\\ \boldsymbol{\nabla}\wedge\mathbf{B} &= \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}. & (41) \end{aligned}$$

Take $\boldsymbol{\nabla}\wedge(...)$ of (38) and use

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E}, \tag{42}$$

where the first term is zero by (40), and $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$. Then we have

$$\nabla^2\mathbf{E} = \boldsymbol{\nabla}\wedge\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(\boldsymbol{\nabla}\wedge\mathbf{B}) = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{43}$$

Thus each (Cartesian) component of $\mathbf{E}$ satisfies a wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{E} = 0, \tag{44}$$

where the wave speed c is given by

$$c^2 = \frac{1}{\epsilon_0\mu_0}. \tag{45}$$

Check that (39) and (41) can be used similarly to show that each component of $\mathbf{B}$ satisfies the same wave equation. In other words, each of $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are propagated as waves of speed c.

The values of the quantities ϵ_{00} and μ_0 appropriate to SI units are fixed by experiment, and these values indicate that

$$c = 3\times10^8 m/s = \text{the speed of light}. \tag{46}$$

Maxwell's equations with the crucial displacement current term, necessary for consistency, can describe electromagnetic wave phenomena across its entire frequency spectrum: see the Table. For waves of frequency ν, measured in hertz, and wavelength λ, measured in metres, $c = \lambda\nu$. Also, in quantum theory, the energy of a quantum of given frequency ν is $E = h\nu$, where h is Planck's constant. (One hertz equals one cycle per second).

Frequency spectrum					
radiation	ν	λ	radiation	ν	λ
γ	10^{19}	10^{-11}	infra-red	10^{14}	10^{-6}
X-rays	10^{18}	10^{-10}	μ-wave	10^{13}	10^{-5}
ultra-violet	10^{16}	10^{-8}	mm	10^{11}	10^{-3}
visible light	10^{15}	10^{-7}	radio	10^{6}	10^{2}

1.7 Discontinuity formulas

Here we collect, for easy reference but *without discussion at this stage* a class of formulas that logically belong together but whose occurrences are scattered throughout several sections of the course material.

Let S be a surface with unit normal $\mathbf{n}$ which separates regions $V_{\pm}$ of space, with $\mathbf{n}$ pointing from S into V_{+}.

a). Let S carry charge density σ per unit area. Let $\mathbf{E}_{\pm}$ denote the electric fields just inside the $V_{\pm}$ sides of S. Then

$$\mathbf{n}\cdot\mathbf{E}|_{-}^{+} = \frac{1}{\epsilon_0}\sigma \tag{47}$$

$$\mathbf{n}\wedge\mathbf{E}|_{-}^{+} = 0. \tag{48}$$

Eq. (47) is proved on the basis of Gauss's theorem in Sec. 2.2. Note eqs. (47) and (48) respectively involve the components of $\mathbf{E}$ normal and tangential ($\mathbf{n}\cdot\mathbf{n}\wedge\mathbf{E}=0$)to the surface S.

b). Let S carry current density $\mathbf{s}$ per unit length (charge crossing unit length in S in unit time). Let $\mathbf{B}_{\pm}$ denote the magnetic fields just inside the $V_{\pm}$ sides of S.

$$\mathbf{n}\cdot\mathbf{B}|_{-}^{+} = 0 \tag{49}$$

$$\mathbf{n}\wedge\mathbf{B}|_{-}^{+} = \mu_0\mathbf{s}. \tag{50}$$

Eq. (49) is proved in the same way as used for (47). Eq. (50) is a consequence of Stokes's theorem, as is (48). A special case of (50) occurs in Sec. 3.3

The correspondence between Maxwell's equations and the discontinuity formulas is clear: drop $\dfrac{\partial}{\partial t}$ terms, and replace $\boldsymbol{\nabla}(...)$ by $\mathbf{n}(...)|_{-}^{+}$. Thus, from (20), we expect that $\mathbf{n}\cdot\mathbf{J}|_{-}^{+}=0$ at a surface of discontinuity, one that may carry surface density of charge.

Force per unit area on S

In case (a), consider only the special case when $\mathbf{E}_{\pm}$ only have normal components $\mathbf{n}\cdot\mathbf{E}_{\pm}=E_{\pm}$. Then the force per unit area on a surface S (carrying surface charge σ) has magnitude

$$\frac{1}{2}\sigma(E_{+}+E_{-}). \tag{51}$$

In case (b), consider only the special case in which $\mathbf{B}_{\pm}$ only have tangential components $B_{\pm}$. Then the force per unit area on a surface S (carrying surface current $\mathbf{s}$) is normal to S, and has magnitude

$$\frac{1}{2}s(B_{+}+B_{-}). \tag{52}$$

We do not prove the results (51) and (52); the most convenient method of proof lies outside the scope of this course.

2 Electrostatics

2.1 Electrostatic potential

Electrostatics is the study of time independent electromagnetic phenomena in the absence of currents and magnetic fields. Then Maxwell's equations are

$$\boldsymbol{\nabla}\wedge\mathbf{E} = 0 \tag{53}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho. \tag{54}$$

Eq. (53) can be satisfied by defining the (electrostatic) potential ϕ by means of

$$\mathbf{E} = -\boldsymbol{\nabla}\phi, \tag{55}$$

so that (54) yields Poisson's equation

$$\boldsymbol{\nabla}^2\phi = -\frac{1}{\epsilon_0}\rho. \tag{56}$$

In this way the study of electrostatics is reduced to the study of a single equation – Poisson's equation. In regions of space where there is no electric charge $\rho = 0$, this reduces to Laplace's equation

$$\boldsymbol{\nabla}^2\phi = 0. \tag{57}$$

ϕ is defined by (55) only to within an additive constant. Usually one choses this constant in such a way that $\phi(\mathbf{r}) \longrightarrow 0$ as $r = |\mathbf{r}| \longrightarrow \infty$. For the point charge q at O, the electric field given by (9) reads

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2}\mathbf{e}_r = -\frac{\partial\phi}{\partial r}\mathbf{e}_r, \quad \mathbf{e}_r = \frac{\mathbf{r}}{r}, \tag{58}$$

and an integration (with constant zero) gives

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r}. \tag{59}$$

Since Poisson's equation is a linear equation for ϕ, the 'superposition principle' applies and tells us that any linear combination (superposition) of solutions is again a solution. A first example of this is

The electric dipole

Consider a system of two point charges $\pm q$, $-q$ at O and $+q$ at $\mathbf{d}$. The superposition principle implies that

$$4\pi\epsilon_0\phi(\mathbf{r}) = q(-\frac{1}{r} + \frac{1}{|\mathbf{r}-\mathbf{d}|}). \tag{60}$$

For all such examples the easiest method of expansion involves the vector statement of Taylor's theorem:

$$f(\mathbf{r}+\mathbf{h}) = f(\mathbf{r}) + \mathbf{h}\cdot\boldsymbol{\nabla}f(\mathbf{r}) + \frac{1}{2}(\mathbf{h}\cdot\boldsymbol{\nabla})^2 f(\mathbf{r}) + \ldots. \tag{61}$$

Here

$$\frac{1}{|\mathbf{r}-\mathbf{d}|} = \frac{1}{r} - \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r} + \ldots. \tag{62}$$

So for $d = |\mathbf{d}|$ small we have

$$4\pi\epsilon_0\phi = -q\mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r}. \tag{63}$$

The electric dipole arises by taking the limits $q \to \infty, d \to 0$ in such a way that qd remains constant, at a finite value $qd = p$. Then $\mathbf{p} = q\mathbf{d}$ defines the dipole moment of the electrical dipole, and its potential is given by

$$4\pi\epsilon_0\phi = -\mathbf{p}\cdot\boldsymbol{\nabla}\frac{1}{r} = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{\mathbf{p}\cdot\mathbf{e}_r}{r^2}. \tag{64}$$

Taking $\mathbf{d} = d\mathbf{k} = (0,0,1)$ in the z-direction, then, working initially in Cartesians so that $\mathbf{p}\cdot\boldsymbol{\nabla} = p\frac{\partial}{\partial z}$, we find that (64) gives us

$$4\pi\epsilon_0\phi = -p\frac{\partial}{\partial z}\frac{1}{r} = -p(-\frac{1}{r^2}\frac{z}{r}) = \frac{pz}{r^3} \quad = \frac{p\cos\theta}{r^2}. \tag{65}$$

In the last step, we used spherical polars with $z = r\cos\theta$.

The electric quadrupole: not lectured

We can easily go further to the linear quadrupole with charges $-q$ at $\pm\mathbf{d}$ and $2q$ at the origin, so that the system has zero total charge and also zero dipole moment. (It looks like a pair of dipoles pointing in oppposite directions.)

$$\begin{aligned}\frac{4\pi\epsilon_0}{q}\phi &= \frac{2}{r} - \frac{1}{|\mathbf{r}+\mathbf{d}|} - \frac{1}{|\mathbf{r}-\mathbf{d}|} \\ &= \frac{2}{r} - [\frac{1}{r} + \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}] - [\frac{1}{r} - \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}] \\ &= -(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}. \end{aligned} \tag{66}$$

Note that this approach gets the cancellation of unwanted terms to happen ahead of their evaluation. Hence

$$4\pi\epsilon_0\phi = -q(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r} = -qd^2\frac{\partial^2}{\partial z^2}\frac{1}{r} = -qd^2\frac{\partial}{\partial z}(-\frac{z}{r^3}) = qd^2(\frac{1}{r^3} - \frac{3z^2}{r^5}). \tag{67}$$

In spherical polars the quadrupole potential is

$$4\pi\epsilon_0\phi = qd^2\frac{1-3\cos^2\theta}{r^3}. \tag{68}$$

We note that the point charge, electric dipole and quadrupole potentials go to zero as r goes to infinity respectively like $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$.

The general charge distribution $\mathcal{D}$

Suppose $\mathcal{D}$ has electric charge density ρ non-zero only throughout some finite subset $\hat{V} \subset V$ =all space. To find the potential due to $\mathcal{D}$, we view it as linear superposition of contributions due to 'elementary charges' $\rho(\mathbf{r}')\,d\tau'$ throughout $\hat{V}$. Then the superposition principle gives

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|}. \tag{69}$$

Since

$$\boldsymbol{\nabla}\frac{1}{|\mathbf{r}-\mathbf{a}|} = -\frac{\mathbf{r}-\mathbf{a}}{|\mathbf{r}-\mathbf{a}|^3}, \quad \mathbf{r}\neq\mathbf{a} \tag{70}$$

we get

$$-\boldsymbol{\nabla}\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V (-\boldsymbol{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}'|})\rho(\mathbf{r}')d\tau' = \frac{1}{4\pi\epsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{E}(\mathbf{r}), \tag{71}$$

consistently with (6) of chapter 1, at least for $\mathbf{r}\notin\hat{V}$. We do not have time to provide the proof, by standard methods in vector calculus, that (69) satisfies Poisson's equation for all $\mathbf{r}\in V$.

Large distance behaviour of (69)

Taking an origin near to or within $\hat{V}$, we want to find how the potential due to $\mathcal{D}$ behaves at large distances r. We will follow the same procedure as above, using Taylor's theorem (61). We find

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \left(\frac{1}{r} - \mathbf{r}'\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\cdot\boldsymbol{\nabla})^2\frac{1}{r}\dots\right)\rho(\mathbf{r}')d\tau'. \tag{72}$$

The leading term of $4\pi\epsilon_0\phi$ (going like $\frac{1}{r}$) is the total charge term, namely

$$\frac{Q}{r}, \quad Q = \int_V \rho(\mathbf{r}')d\tau', \tag{73}$$

unless $Q = 0$. In the latter case the leading term (going like $\frac{1}{r^2}$) is the dipole term

$$-\left(\int_V \mathbf{r}'\rho(\mathbf{r}')d\tau'\right)\cdot\boldsymbol{\nabla}\frac{1}{r} = -\mathbf{P}\cdot\boldsymbol{\nabla}\frac{1}{r}, \tag{74}$$

where the dipole moment of the distribution is

$$\mathbf{P} = \int_V \mathbf{r}'\rho(\mathbf{r}')d\tau', \tag{75}$$

unless of course $\mathbf{P} = 0$, in which case the leading term is a quadrupole type term which goes like $\frac{1}{r^3}$ ....

Uniqueness

Suppose we are given a charge distribution $\rho(\mathbf{r})$ throughout a fixed spatial volume V, then Poisson's equation in V has a unique solution provided that, on $S = \partial V$, either

(i) (Dirichlet boundary conditions) $\phi(\mathbf{r})$ is specified for all $\mathbf{r} \in S$,

or

(ii) (Neumann boundary conditions) $\dfrac{\partial\phi}{\partial n} = \mathbf{n}\cdot\boldsymbol{\nabla}\phi(\mathbf{r}) = -\mathbf{n}\cdot\mathbf{E}(\mathbf{r})$ is specified for all $\mathbf{r} \in S$.

We will see soon that the latter option corresponds to specifying the surface density of charge on S.

For the case of $\mathcal{D}$ above, with a choice of origin near $\hat{V}$, it follows that (69) is the unique solution of Poisson's equation which satisfies the (Dirichlet) boundary condition that $\phi \to 0$ as $r \to \infty$.

Field lines and equipotentials

We mention a way of gaining some insight into the nature of the electric field surrounding a system of charges.

One draws the field lines of $\mathbf{E}$ for the system. A field line here is a line at each of whose points $\mathbf{E}$ is tangent to the line.

Also one draws on the same diagram the equipotentials of the system. These are surfaces ϕ =constant. As $\mathbf{E} = -\boldsymbol{\nabla}\phi$, and $\boldsymbol{\nabla}\phi$ is everywhere normal to such surfaces, it follows that the field lines cut the equipotentials at right angles.

2.2 Gauss's theorem and the calculation of electric fields

In Sec. 1.5 we proved Gauss's theorem

$$\frac{1}{\epsilon_0}Q = \int_S \mathbf{E}\cdot\mathbf{dS}, \tag{76}$$

where

$$Q = \int_V \rho d\tau, \tag{77}$$

is the total charge contained in the spatial volume V, $\partial V = S$. We now apply it to the calculation of the electric fields of simple systems of charge.

a) The point charge q at the origin has been treated in Sec 1.1.

b) Line charge lying along the z-axis with uniform (line) density of charge η (Coulombs) per unit length. Let S be the closed surface of a right circular cylinder of unit length coaxial with the line charge. By symmetry, it is clear that $\mathbf{E}$ is radial, so $\mathbf{E}\cdot\mathbf{n} = 0$ on the ends of S. In fact $\mathbf{E}(\mathbf{r}) = E(s)\mathbf{e}_s$ where s and $\mathbf{e}_s$ are the radial coordinate of cylindrical polars and its associated unit vector. Thus Gauss gives

$$E\ 2\pi s = \frac{1}{\epsilon_0}\eta, \quad \mathbf{E}(s) = \frac{\eta}{2\pi\epsilon_0}\frac{1}{s}\mathbf{e}_s. \tag{78}$$

This corresponds to a potential given by

$$2\pi\epsilon_0\phi = -\eta \log\frac{s}{s_0}. \tag{79}$$

In this example, $\phi(s)$ does not go to zero as $s \to \infty$, so we were forced to demand that $\phi = 0$ for some fixed but arbitrary value s_0 of s.

To check that (79) is correct, use

$$-\boldsymbol{\nabla}\phi = -\mathbf{e}_s\frac{\partial\phi}{\partial s} \tag{80}$$

c) Plane sheet P occupying the plane $z = 0$, carrying uniform charge density σ per unit area.

Here we use the 'Gaussian pillbox': a cylinder of cross-sectional area A, with axis $\mathbf{k} = (0,0,1)$, with plane ends at $z = h$ and $z = -h$. By symmetry $\mathbf{E}$ is perpendicular to P. Above P we have $\mathbf{E} = E\mathbf{k}$ and below $\mathbf{E} = -E\mathbf{k}$ for some $E = E(h)$. This time $\mathbf{E}\cdot\mathbf{dS}$ is zero on the curved sides of the pill-box, and Gauss gives

$$EA - (-E)A = \frac{\sigma\, A}{\epsilon_0}, \quad E = \frac{1}{2\epsilon_0}\sigma, \quad \text{independent of } h. \tag{81}$$

d) Parallel plane sheets in the planes $z = 0$ and $z = a$, carrying uniform distributions of charge respectively of charge with surface densities $\pm\sigma$ (Coulombs) per unit area. Using the result of c) twice and the principle of superposition, we find that

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\mathbf{k}, \quad \mathbf{k} = (0,0,1), \tag{82}$$

in the spatial region between the plates and zero outside.

Writing $\mathbf{E} = -\frac{\partial\phi}{\partial z}\mathbf{k}$, we get $\phi = \phi_0 - \frac{\sigma}{\epsilon_0}z$, where ϕ_0 is the potential of the $z = 0$ sheet. If the $z = a$ sheet has potential ϕ_a, then the potential difference between the sheets is $\phi_0 - \phi_a = \frac{\sigma}{\epsilon_0}a$.

e) Spherical shell, centre at O, radius r', uniform charge density σ per unit area, and thus total charge $Q = 4\pi r'^2\sigma$. By symmetry, as for a point charge at O, we have $\mathbf{E} = E(r)\mathbf{e}_r$.

To apply Gauss's theorem, take spheres of radius r, concentric with the shell. Let these have surfaces S_1 and S_2, in the cases (i) $r > r'$ and (ii) $r < r'$

In case (i):

$$\begin{aligned}\int_{S_1} \mathbf{E}\cdot\mathbf{dS} = \int_{S_1} E(r)\mathbf{e}_r\cdot\mathbf{e}_r dS &= \frac{1}{\epsilon_0}Q \\ 4\pi r^2 E(r) &= \frac{1}{\epsilon_0}Q, \quad E(r) = \frac{\sigma}{\epsilon_0}\frac{r'^2}{r^2}.\end{aligned} \tag{83}$$

For case (ii), we have $E(r) = 0$, since there is no charge in the volume V_2.

It is to be noted that the result (78) is the same (for $r > r'$) as applies to a point charge Q situated at the origin.

Check that $E = \mathbf{E}\cdot\mathbf{e}_r$, the normal component of $\mathbf{E}$, has discontinuity $\frac{1}{\epsilon_0}\sigma$ at $r = r'$.

f) Sphere of radius R carrying uniform charge of density ρ (Coulombs) per unit volume, and thus total charge $Q = \frac{4\pi}{3}R^3\rho$.

For $r > R$ by superposition of shells and the result of e), we learn that the potential is the same as it would be if we had a point charge Q at the origin.

$$\mathbf{E}(\mathbf{r}) = E_1(r)\mathbf{e}_r, \quad E_1(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}. \tag{84}$$

For $r < R$, applying Gauss to a sphere S_2 centre the origin of radius r, only the charge inside S_2 is relevant, and we have

$$\mathbf{E}(\mathbf{r}) = E_2(r)\mathbf{e}_r, \quad E_2(r)\,4\pi r^2 = \frac{1}{\epsilon_0}\rho\,\frac{4\pi}{3}r^3, \tag{85}$$

so that inside the charge distribution

$$E_2(r) = \frac{Qr}{4\pi\epsilon_0 R^3}. \tag{86}$$

We have obtained (86) by direct application of Gauss, but we could otherwise have found it from e) by a suitable application of the superposition principle.

Note that $E(r)$, the normal (and here only) component of $\mathbf{E}$, is continuous at $r = R$.

We can use $\mathbf{E} = -\boldsymbol{\nabla}\phi = -\mathbf{e}_r\dfrac{\partial\phi}{\partial r}$ to determine the potentials ϕ_1 outside, and ϕ_2 inside, the charge distribution.

$$\begin{aligned}-\frac{\partial\phi_1}{\partial r} &= \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \quad\Rightarrow\quad \phi_1 = \frac{Q}{4\pi\epsilon_0}\frac{1}{r} + A \\ -\frac{\partial\phi_2}{\partial r} &= \frac{Qr}{4\pi\epsilon_0 R^3} \quad\Rightarrow\quad \phi_2 = -\frac{Qr^2}{8\pi\epsilon_0 R^3} + B.\end{aligned} \tag{87}$$

Here A and B are constants of integration. Demanding that $\phi \to 0$ as $r \to \infty$, we look at ϕ_1 and require $A = 0$. To find B, we use the fact that ϕ is continuous at $r = R$. This leads to

$$\phi_2 = \frac{Q}{8\pi\epsilon_0 R^3}(3R^2 - r^2). \tag{88}$$

g) The discontinuity law at a surface carrying surface charge.

Suppose a surface S with normal $\mathbf{n}$ carrying charge of uniform charge density σ per unit area, separates regions 1 and 2 of empty space, with $\mathbf{n}$ pointing into 2. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric fields in regions 1 and 2.

Use Gauss with a Gaussian pillbox of very small height, and cross sectional area A, with the end with normal $\mathbf{n}$ just inside 2 and the other end with normal $-\mathbf{n}$, just inside 1. In fact we take the height so small that the curved sides of the box contribute negligibly to the surface integral of the theorem. Then

$$[\mathbf{n}\cdot\mathbf{E}_2 + (-\mathbf{n})\cdot\mathbf{E}_1]A = \frac{\sigma A}{\epsilon_0}, \quad \mathbf{n}\cdot\mathbf{E}|_-^+ = \frac{1}{\epsilon_0}\sigma, \tag{89}$$

as stated in Sec. 1.7.

See that examples c), d), e), and f) conform to this, there being no surface charge present in f).

Solutions of Laplace's equations

In spherical polars (r,θ,ϕ), the general solution of Laplace's equations with spherical symmetry (with no dependence on θ and ϕ) is

$$\phi = a + \frac{b}{r}. \tag{90}$$

Next we have solutions, like the dipole potential, $\propto \cos\theta$,

$$\phi = -Er\cos\theta + \frac{c}{r^2}\cos\theta = \phi_1 + \phi_2. \tag{91}$$

Here the first term gives an electric field $\mathbf{E} = -\boldsymbol{\nabla}\phi_1 = -\frac{\partial\phi_1}{\partial z}\mathbf{k} = E\mathbf{k}$ of constant magnitude in the z-direction.

In cylindrical polars (s,ϕ,z), the general solution of Laplace's equations with cylindrical symmetry is

$$\phi = a + b\ln s. \tag{92}$$

2.3 Perfect conductors

In a perfect conductor any movable charges present are free to move within it under an applied electric field without resistance. In electrostatics, we deal with situations in which all such movable charges have reached positions of equilibrium. In particular there is no current flow, the atoms of the material keep their conduction electrons and are neutral.

Consider then a perfect conductor $\mathcal{C}$ with surface S, with perfectly non-conducting empty space (the vacuum) outside, to which some non-zero total charge has been supplied.

We shall see in Sec. 3.1 that inside $\mathcal{C}$ we must have $\mathbf{E}=0$, and hence $\rho = 0$. Thus it follows that the charge supplied must reside on the surface S of $\mathcal{C}$. Further $\mathbf{E} = E\mathbf{n}$ on S, else charges would be able to move along S. Thus S is an equipotential of constant ϕ, since $\mathbf{E} = -\boldsymbol{\nabla}\phi$ is normal to it. Also, because $\mathbf{E}=0$ inside S, ϕ is constant throughout the interior of $\mathcal{C}$, with a value equal to the surface equipotential value. Finally the charge σ per unit area on S follows from g) of Sec. 2.2. This gives

$$\frac{1}{\epsilon_0}\sigma = \mathbf{n}\cdot\mathbf{E}|_-^+ = \mathbf{n}\cdot\mathbf{E} = E \tag{93}$$

This uses the fact that $\mathbf{E}=0$ inside $\mathcal{C}$, (*i.e.* on the minus side of the surface S of $\mathcal{C}$).

The Force on a charged conductor

We do not have time to give a proof, but note the force per unit area exerted on the surface of a perfect conductor carrying charge per unit area σ is given by

$$F = \frac{1}{2\epsilon_0}\sigma^2. \tag{94}$$

This is a special case of the result (51) of Sec. 1.7.

2.4 Electrostatic energy

The potential energy (PE) of a point charge q at $\mathbf{r}$ in an electric field of potential $\phi(\mathbf{r})$ is the work that must be done on q to bring it from infinity (where $\phi = 0$) to $\mathbf{r}$

$$PE = q\phi(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r}, \quad \mathbf{F} = q\mathbf{E}. \tag{95}$$

Consider a system of point charges $q_i, \quad i = 1, 2, \ldots, n$, bringing them from infinity to their final positions in order, doing work

$$\begin{aligned}
\text{on} \quad q_1; \quad W_1 &= 0 \\
\text{on} \quad q_2; \quad W_2 &= \frac{q_2}{4\pi\epsilon_0}\frac{q_1}{r_{12}} \\
\text{on} \quad q_3; \quad W_3 &= \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) \\
\text{on} \quad q_i; \quad W_i &= \frac{q_i}{4\pi\epsilon_0}\sum_{j<i}\frac{q_j}{r_{ji}} \\
W = \sum_{i=1}^{n} W_i &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{4\pi\epsilon_0}\frac{q_i\, q_j}{r_{ij}}.
\end{aligned} \tag{96}$$

Here $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \quad r_{ij} = |\mathbf{r}_{ij}|$, and $\sum_{i=1}^{n}\sum_{j<i} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}$. Thus W by construction gives the electrostatic energy of the system.

But the potential at q_i due to all the other charges is

$$\phi_i = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{r_{ij}}, \tag{97}$$

so that

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\phi_i. \tag{98}$$

The corresponding result for a continuous distribution of charge of charge density $\rho(\mathbf{r})$ in volume V then is

$$\begin{aligned}
W &= \frac{1}{2}\int_V \rho(\mathbf{r})\phi(\mathbf{r})d\tau \\
&= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau\, d\tau'.
\end{aligned} \tag{99}$$

If there are conductors $\mathcal{C}_i$ with charges Q_i at potentials ϕ_i, then the contribution which they make to W is given by

$$\frac{1}{2}\sum_i \int_{S_i}\sigma_i\phi_i dS_i = \frac{1}{2}\sum_i \phi_i\int_{S_i}\sigma_i dS_i = \frac{1}{2}\sum_i \phi_i Q_i. \tag{100}$$

(Recall that the potential is constant on a conductor).

Field energy in electrostatics

Given a charge distribution $\rho(\mathbf{r}')$ distributed over a finite volume $\hat{V}$ and a set of conductors all in some finite region of space in which an origin is taken. Let V be all space bounded by a sphere S at infinity, but excluding the interiors of the conductors.

Then

$$W = \frac{1}{2}\int_V \rho\phi d\tau + \frac{1}{2}\sum_i Q_i\phi_i. \tag{101}$$

Use

$$\begin{aligned}\rho\phi &= \epsilon_0 \phi \boldsymbol{\nabla}\cdot\mathbf{E} \\ &= \epsilon_0[\boldsymbol{\nabla}\cdot(\phi\mathbf{E}) - \mathbf{E}\cdot\boldsymbol{\nabla}\phi] \\ &= \epsilon_0\boldsymbol{\nabla}\cdot(\phi\mathbf{E}) + \epsilon_0\mathbf{E}^2. \end{aligned} \tag{102}$$

Then W is given by

$$\frac{1}{2}\epsilon_0[\int_V \mathbf{E}^2 d\tau + \int_S \phi\mathbf{E}\cdot\mathbf{dS} + \sum_i \int_{\mathcal{C}_i} \phi\mathbf{E}\cdot\mathbf{dS}_i] + \frac{1}{2}\sum_i Q_i\phi_i. \tag{103}$$

As $\phi = 0$ on S (at infinity) the second term of (103) is zero. In the third term of (103), the divergence theorem dictates that $\mathbf{dS}_i = -\mathbf{n}dS_i$ points into $\mathcal{C}_i$, and so, in this term, we have

$$-\epsilon_0\int_{\mathcal{C}_i}\phi\mathbf{n}\cdot\mathbf{E}dS_i = -\epsilon_0\phi_i\int_{\mathcal{C}_i}\mathbf{n}\cdot\mathbf{E}dS_i = -\phi_i\int_{\mathcal{C}_i}\sigma_i dS_i = -\phi_i Q_i. \tag{104}$$

It follows that the third and the fourth terms of (103) cancel. And so, for the energy of the electrostatic field, we have the important result

$$W = \frac{1}{2}\epsilon_0\int_V \mathbf{E}^2 d\tau. \tag{105}$$

We note this involves an integral over all of V, including the regions unoccupied by charge, whereas the first term of (101) is really an integral over the region $\hat{V}\subset V$ occupied by charge.

2.5 Capacitors and capacitance

A pair of conductors carrying charges $\pm Q$ constitute a capacitor (or a condenser). Since their potentials are proportional to Q, the same applies to their potential difference $V = \phi_1 - \phi_2$.

Therefore we define the capacitance C of the capacitor by

$$V = \frac{1}{C}Q. \tag{106}$$

It turns out always to be a constant that depends on the configuration of the two conductors.

a) Parallel-plate capacitor.

The field lines are mainly straight lines perepndicular to the plates. We assume the distance a between the plates is small on a scale set by the area A of the plates. Thus we may neglect ' edge effects', so called because the electric field lines near to the edges of the plates bulge out from between the plates.

¿From d) of Sec. 2.2, we know that $\mathbf{E} = E\mathbf{k}$, $\quad E = \dfrac{\sigma}{\epsilon_0}$ between the plates, with $E = 0$ elsewhere. Here $\mathbf{k} = (0, 0, 1)$. Hence

$$-\frac{d\phi}{dz} = E \Rightarrow \phi = -Ez + c. \tag{107}$$

If $\phi = \phi_1$ at $z = 0$, then $c = \phi_1$, and then $\phi = \phi_2$ at $z = a$ gives

$$\phi_2 = -Ea + \phi_1, \quad \text{and} \quad V = \phi_1 - \phi_2 = aE = \frac{a\sigma}{\epsilon_0} = \frac{aQ}{\epsilon_0 A}. \tag{108}$$

So

$$C = \frac{A\epsilon_0}{a}. \tag{109}$$

The energy of the capacitor is given now by (98), so that

$$W = \frac{1}{2}\sum_i q_i\phi_i = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}. \tag{110}$$

But the energy can also be calculated from the field energy expression (105), which gives

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{A\epsilon_0}{2}\int_0^a (\frac{\sigma}{\epsilon_0})^2 dz = \frac{\sigma^2 Aa}{2\epsilon_0} = \frac{1}{2}\frac{Q^2}{C}. \tag{111}$$

b) Concentric spheres S_1 and S_2 of radii a and $b > a$, carrying charges Q and $-Q$. Take $\phi = 0$ at $r = b$ and $\phi = V$ at $r = a$. ¿From previous studies we know that for $r \in \{a \leq r \leq b\}$ (outside S_1 and inside S_2) we have

$$4\pi\epsilon_0 E = -4\pi\epsilon_0\frac{\partial\phi}{\partial r} = \frac{Q}{r^2}, \tag{112}$$

and

$$4\pi\epsilon_0\phi = \frac{Q}{r} - \frac{Q}{b}. \tag{113}$$

Hence

$$4\pi\epsilon_0 V = Q(\frac{1}{a} - \frac{1}{b}) \tag{114}$$

and

$$C = \frac{4\pi\epsilon_0 a\, b}{(b-a)}. \tag{115}$$