Mathematical Tripos 2004, Part IB : Electromagnetism, Lent 2004
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3 Steady electric currents and magnetism

3.1 Steady current flow

Here we study steady current flow in conducting material. This is governed by Maxwell's equations without $\dfrac{\partial}{\partial t}$ terms, so that we have

$$\nabla\wedge\mathbf{B} = \mu_0\mathbf{J}, \quad \nabla\wedge\mathbf{E} = 0, \tag{1}$$

together with the experimental law, valid for simple conductors, but not, for example, for non-isotropic materials such as crystalline material,

$$\mathbf{J} = \sigma\mathbf{E}, \tag{2}$$

where σ is the conductivity of the material.

(Both conductivity and surface charge are normally denoted by the same symbol σ. We seldom have contexts in which both arise.)

Note that (1) implies

$$\nabla\cdot\mathbf{J} = 0. \tag{3}$$

This agrees the continuity equation, eq. (20) of chapter one, as $\dfrac{\partial\rho}{\partial t} = 0$ applies here. Eq. (2) also implies

$$\nabla\cdot\mathbf{E} = 0, \tag{4}$$

and hence also $\rho = 0$ within the material. This makes sense in contexts such as current flowing in copper wires in which electrons flow through a background of positively charge ions, so that it is reasonable to suppose that $\rho = 0$ for the total charge density of the material, electrons plus ions.

We have a remark here promised in Sec. 2.3 which talks about perfect conductors for which the conductivity σ goes to infinity. In order for finite currents ($|\mathbf{J}|$ finite) to flow in such material, it is necessary that $|\mathbf{E}|$ goes to zero, so that also ρ goes to zero.

In this section, we are concerned only with current flow. In later sections of this chapter, we study the magnetic fields that arise from the (time-independent) flow of electric currents.

Consider steady current flow in regions of conducting material, outside of batteries.

This is governed by the equations

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\wedge\mathbf{E} = 0, \tag{5}$$

together with the experimental law (2).

If we set $\mathbf{E} = -\nabla\phi$ then the flow is governed by the single equation, Laplace's equation, plus (2), so that we can solve problems of steady current flow by finding $\phi, \mathbf{E}, \mathbf{J}$ in turn.

We might ask: can we obtain an understanding of the elementary form

$$V = IR \tag{6}$$

of Ohm's law, relating the potential difference across the ends of a conductor to the current that flows within it?

We do this here for a simple example; there are two others in Problem Set 2.

Uniform current enters the plate of uniform thickness δ shown in the diagram. In cylindrical polars, (with polar angle called θ since ϕ is reserved here for the potential), we have the solution

$$\phi = d - c\theta, \quad c,\ d \quad \text{constants}, \tag{7}$$

of Laplace's equation, so that the potential difference (PD) between AB and CD is $V = c\alpha$. Hence

$$\mathbf{E} = -\frac{1}{s}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta = \frac{c}{s}\mathbf{e}_\theta, \tag{8}$$

and the lines of $\mathbf{E}$ and of $\mathbf{J}$ are arcs of circles centred on O, as shown. Also

$$\mathbf{J} = \sigma\mathbf{E} = \frac{\sigma c}{s}\mathbf{e}_\theta = \frac{\sigma V}{\alpha s}\mathbf{e}_\theta \tag{9}$$

so that the total current entering at AB (which of course equals the current leaving at CD) is

$$I = \int_{AB} \mathbf{J}\cdot\mathbf{dS} \ = \ \frac{\sigma V\delta}{\alpha}\int_{s_1}^{s_2}\frac{1}{s}ds = \frac{\sigma V\delta}{\alpha}\ln\frac{s_2}{s_1}, \tag{10}$$

where we used $\mathbf{dS} = \mathbf{e}_\theta ds\delta$, and (9). This is indeed of the form (6) of Ohm's law, with

$$R = \frac{\alpha}{\sigma\delta\ln(s_2/s_1)}. \tag{11}$$

So resistance is inversely proportional to conductivity σ, and, like capacitance, depends on the geometry of the current flow set-up.

Generation of heat by steady current flow

Consider the tube of flow shown, *i.e.* the cylinder whose sides are lines of $\mathbf{E}$ and $\mathbf{J}$ and whose ends are equipotentials. Current of density $\mathbf{J}$ enters at the end A where the potential is ϕ_A and leaves at B where the potential is $\phi_B < \phi_A$. The potential difference is

$$V = \phi_A - \phi_B = -\delta\mathbf{r}\cdot\boldsymbol{\nabla}\phi = \delta r\, E. \tag{12}$$

In unit time charge $J\delta S$ enters the tube at A in unit time and leaves at B. The work done on this charge moving it through the potential difference V in unit time is

$$(J\delta S)V = (J\delta S)(E\delta r) = JE(\delta S\ \delta r) = (\mathbf{J}\cdot\mathbf{E})\delta\tau. \tag{13}$$

This work done corresponds to the conversion of electrical energy into heat, *i.e.* to the loss of electrical energy. The energy loss per unit time in volume τ, with surface S is

$$W = \int_\tau \mathbf{J}\cdot\mathbf{E}d\tau = -\int_\tau \mathbf{J}\cdot\boldsymbol{\nabla}\phi d\tau. \tag{14}$$

We use

$$\mathbf{J}\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\cdot(\phi\mathbf{J}) - \phi(\boldsymbol{\nabla}\cdot\mathbf{J}), \tag{15}$$

where the second term is zero owing to (3), and the first term allows us to apply the divergence theorem to (14). We obtain

$$W = -\int_S \phi\mathbf{n}\cdot\mathbf{J}dS, \tag{16}$$

where $\mathbf{n}$ is the unit normal on S pointing out of τ.

Consider a conductor with current entering it and leaving it at ends S_1 and S_2, which are equipotentials of potentials ϕ_1 and ϕ_2. Then, remembering that the $\mathbf{n}$ of (16) for S_1 is the negative of $\mathbf{n}_1$ in the diagram, we have from (16)

$$\begin{aligned} W &= (\phi_1-\phi_2)I, \quad I = \int_{S_1}\mathbf{n}_1\cdot\mathbf{J}dS = \int_{S_2}\mathbf{n}_2\cdot\mathbf{J}dS \\ &= VI, \end{aligned} \tag{17}$$

where V is the potential difference between the ends. Using the elementary form (6) of Ohm's law, we have shown that the energy generation per unit time in a conductor of resistance R through which flows a current I is

$$W = RI^2. \tag{18}$$

This is a formula familiar from elementary studies for the energy dissipated in unit time as heat.

3.2 Magnetostatics

This deals with steady currents and the associated (time independent) magnetic fields. It is governed by the equations

$$\boldsymbol{\nabla}\wedge\mathbf{B} = \mu_0\mathbf{J}, \quad (\Rightarrow \boldsymbol{\nabla}\cdot\mathbf{J}=0) \tag{19}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{20}$$

Eq. (20) is automatically satisfied when the vector potential $\mathbf{A}$ is introduced via

$$\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}, \tag{21}$$

since

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\wedge\mathbf{A}) = \partial_i\epsilon_{ijk}\partial_jA_k = \boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{22}$$

For given $\mathbf{B}$ however (21) does not determine $\mathbf{A}$ uniquely, because we can transform the vector potential according to

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{23}$$

where χ is an arbitrary scalar field. Since

$$\boldsymbol{\nabla}\wedge\mathbf{A}' = \boldsymbol{\nabla}\wedge\mathbf{A} + \boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\chi = \boldsymbol{\nabla}\wedge\mathbf{A} = \mathbf{B}, \tag{24}$$

the transformed vector potential serves our needs just as well as does $\mathbf{A}$.

In fact we can make use of (23) to impose a simplifying condition on the vector potentials we use in practice. Suppose we have found some $\mathbf{A}$ which yields the required $\mathbf{B}$ via (21), and is such that $\boldsymbol{\nabla}\cdot\mathbf{A} = \psi$, where ψ is a scalar field, calculable, as is obvious, from $\mathbf{A}$. We shall pass by means of (23) to a vector potential $\mathbf{A}'$ such that

$$\boldsymbol{\nabla}\cdot\mathbf{A}' = 0. \tag{25}$$

This can always be done, since (25) implies

$$\begin{aligned} 0 &= \boldsymbol{\nabla}\cdot\mathbf{A} + \boldsymbol{\nabla}^2\chi \\ &= \psi + \boldsymbol{\nabla}^2\chi, \end{aligned} \tag{26}$$

which is an equation of Poisson type for which a (particular integral) solution for χ in terms of ψ can always be found.

In what follows, we therefore assume that we can deal with vector potentials $\mathbf{A}$ which obey

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{27}$$

[**Some language**: Eq. (23) is called a gauge transformation, the condition (27) is called a gauge condition, and the physical theory is said to be gauge-invariant, because it depends only on $\mathbf{B}$, and not on the gauge condition that has been used]

Return now to (19). Since

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A}, \tag{28}$$

(19) reduces, with the aid crucially of our gauge condition (27), to

$$\boldsymbol{\nabla}^2\mathbf{A} = -\mu_0\,\mathbf{J}. \tag{29}$$

In Cartesian coordinates this reads as

$$\boldsymbol{\nabla}^2 A_k = -\mu_0 J_k \qquad (k = 1,2,3), \tag{30}$$

which, for each k, is of Poisson type, so that as in electrostatics, we can write down the solution

$$\begin{aligned} A_k(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \frac{J_k(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\ \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'. \end{aligned} \tag{31}$$

Since it is not obvious that the expression (31) for $\mathbf{A}$ satisfies (27), we ought to prove that it does. When this is done, it follows that

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\wedge\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\wedge(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau', \tag{32}$$

satisfies (19). In calculating $\mathbf{B}$, note that $\boldsymbol{\nabla}\wedge$ acts only on the $\mathbf{r}$ variable, found only in the denominator factor of expression (31) for $\mathbf{A}$, so that we may use eq. (64) of Sec. 2.1 to finish the verification.

Consider a current of density $\mathbf{J}$ flowing in an element $\delta\mathbf{r}$ of a very thin wire of cross-sectional area A. Then $\mathbf{J}\delta V = J\,(A\delta\mathbf{r}) = (JA)\,\delta\mathbf{r} = I\delta\mathbf{r}$. Neglecting the thickness of the wire, we can write, for the vector-potential and the magnetic field due to a wire which carries a current I and takes the form of a simple curve C, the expressions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{\mathbf{dr}'}{|\mathbf{r}-\mathbf{r}'|} \tag{33}$$

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\int_C \frac{(\mathbf{r}-\mathbf{r}')\wedge\mathbf{dr}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{34}$$

The results (32) and (34) for $\mathbf{B}$ are each often referred to the Biot-Savart law.

Proof that (31) satisfies (27).

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \quad (\boldsymbol{\nabla}\text{ acts on }\mathbf{r}\text{ and not on }\mathbf{r}')$$

$$
\begin{aligned}
&= \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_V \left[\boldsymbol{\nabla}'\cdot\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{J}(\mathbf{r}')\right)-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}')\right]d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_S \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{n}'\cdot\mathbf{J}(\mathbf{r}')dS'. \qquad (35)
\end{aligned}
$$

Here V is all space, but if we suppose that a physical current distribution occupies a finite volume $\hat{V}\subset V$ near the origin, then $\mathbf{J}(\mathbf{r}')=0$ on S and the proof is complete.

Note the use of a now well-known identity for $\boldsymbol{\nabla}\cdot(\phi\mathbf{F})$ in the third line, $\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}')=0$ in the fourth line, and finally the ubiquitous divergence theorem.

[**Care with $\boldsymbol{\nabla}^2\mathbf{F}$ for a vector field $\mathbf{F}$** may be needed. There is no problem in Cartesians, and hence probably not in the material of this course:

$$(\boldsymbol{\nabla}^2\mathbf{F})_k = (\partial_j\partial_j)F_k \qquad (36)$$

where $\nabla^2=\partial_j\partial_j$ is the usual expression used in Laplace's equation. In other coordinate systems, where the unit basis vectors are themselves coordinate dependent, $(\boldsymbol{\nabla}^2\mathbf{F})_\alpha$, the component of the vector $\boldsymbol{\nabla}^2\mathbf{F}$ along the unit vector $\mathbf{e}_\alpha$, is no longer given by $(\nabla^2)\mathbf{F}_\alpha$. The correct result however follows from use of $\boldsymbol{\nabla}^2\mathbf{F}=-\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{F})+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F})$ where each of the two terms on the right is calculable by two well-defined steps in any system of orthogonal curvilinear co-ordinates.]

3.3 Magnetic fields of simple current distributions

To calculate these one may use Ampère's law, the Biot-Savart law or perhaps first calculate $\mathbf{A}$ from (31) or (33).

a) Infinite straight wire carrying current I

Take the z-axis along the wire, take O in the xy-plane throught the point P, and calculate $\mathbf{B}$ at P, $\mathbf{r}=\vec{OP}$ using Biot-Savart. Using cylindrical polars, (s,ϕ,z), we have

$$\mathbf{r}=s\mathbf{e}_s,\quad \mathbf{r}'=z'\mathbf{k},\quad \mathbf{dr}'=dz'\mathbf{k},\quad |\mathbf{r}-\mathbf{r}'|=(s^2+z'^2)^{1/2}. \qquad (37)$$

Now $-(\mathbf{r}-\mathbf{r}')\wedge\mathbf{dr}'=s\,dz'\mathbf{e}_\phi$ so that we have proved that $\mathbf{B}$ is everywhere in the direction of $\mathbf{e}_\phi$. Hence, from (34)

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{s\,dz'}{(s^2+z'^2)^{3/2}}\mathbf{e}_\phi$$

$$
\begin{aligned}
&= \frac{\mu_0 I}{4\pi s}\int_{-\pi/2}^{\pi/2} \cos\alpha \, d\alpha \, \mathbf{e}_\phi, \quad (z' = s\tan\alpha) \\
&= \frac{\mu_0 I}{2\pi s}\mathbf{e}_\phi. \qquad (38)
\end{aligned}
$$

We got the same answer in Sec. 1.4, arguing there that $\mathbf{B} = B(s)\mathbf{e}_\phi$ by 'symmetry considerations'.

b) Long solenoid

This is a continuous wire carrying current I wound round a very long right circular cylinder, so long that end effects can be ignored. Assume there are N turns of wire per unit length, with N large, wound in a spiral of very small pitch, so that we can regard the cylindrical surface as carrying a surface current. Use cylindrical polars (r, ϕ, z), with z-axis at the axis of the cylinder. Then $\mathbf{s} = NI\mathbf{e}_\phi$ gives the current density, *i.e.* the current per unit length, measuring the charge crossing unit length in unit time. Note that **we called the radial coordinate of cylindrical polars** r here because the symbol s denotes the magnitude of the surface current.

Since $|\mathbf{B}|$ is clearly independent of both z and ϕ, we take $\mathbf{B}$ of the form

$$
\mathbf{B} = B_z(r)\mathbf{k}, \quad \mathbf{k} = (0, 0, 1). \qquad (39)
$$

This satisfies $\boldsymbol{\nabla}.\mathbf{B} = 0$. Also $\boldsymbol{\nabla}\wedge\mathbf{B} = 0$, which holds where there is no (volume) density of current, implies

$$
\frac{\partial B_z}{\partial r} = 0, \quad \text{so that} \quad B_z = \text{constant}. \qquad (40)
$$

(The cylindrical polar coordinate detail of each of these statements should be checked.)

Outside the cylinder this constant is zero, because $|\mathbf{B}| = 0$ for infinite r. To find $|\mathbf{B}|$ inside the cylinder use the rectangular contour C shown in the diagram on P21. Only the vertical line inside the solenoid contributes to $\oint \mathbf{dr}\cdot\mathbf{B}$, so that Ampère leads to

$$
B_z\, z = \mu_0 NIz, \quad B_z = \mu_0 NI, \quad \mathbf{B} = \mu_0 NI\mathbf{k}. \qquad (41)
$$

The answer obtained here illustrates the general discontinuity law given as eq. (50) of Sec. 1.7, and proved in Sec. 3.8

$$
\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0\, \mathbf{s}, \qquad (42)
$$

at a surface of discontinuity carrying a surface current density $\mathbf{s}$ per unit length. We have $\mathbf{n}\wedge\mathbf{B}|^+ = 0$, and

$$
\mathbf{n}\wedge\mathbf{B}|_- = (-)\mathbf{e}_r\wedge(\mu_0\, NI\mathbf{k}) = \mu_0\, NI\mathbf{e}_\phi = \mu_0\mathbf{s}. \qquad (43)
$$

c) Long cylindrical conductor

Consider current, flowing in a long right circular cylinder and distributed uniformly over its circular cross-section, of area $A = \pi a^2$, so that

$$
\mathbf{J} = J\mathbf{k}, \quad \pi a^2 J = I, \quad \mathbf{k} = (0, 0, 1). \qquad (44)
$$

Assume that magnetic fields can be calculated within the conducting material by the same formulas as apply in the vacuum or free-space. This is a good approximation for good conductors, which do have similar magnetic properties to free-space.

Use cylindrical polars (s, ϕ, z) with z-axis along the axis of the conductor. By symmetry $\mathbf{B} = B(s)\mathbf{e}_\phi$, and we apply Ampère to horizontal circles centred on the z-axis for (i) $s > a$ and $s < a$.

$$\text{outside} \quad 2\pi s B = \mu_0 I, \quad B = \frac{\mu_0 I}{2\pi s} \tag{45}$$

$$\text{inside} \quad 2\pi s B = \mu_0 \pi s^2 J, \quad B = \frac{\mu_0 I s}{2\pi a^2} = \frac{\mu_0 I s}{2}. \tag{46}$$

Note that outside the conductor the magnetic field is the same as for a very thin wire, as in example **a**).

Note also that here there is no surface current, and hence we expect

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = 0. \tag{47}$$

Here $\mathbf{n}\wedge\mathbf{B} = \mathbf{e}_s\wedge B(s)\mathbf{e}_\phi = B(s)\mathbf{k}$ and continuity of the tangential component of $\mathbf{B}$ at $s = a$ follows (45) and (46).

3.4 The large distance expansion of the vector potential

Consider the formula (31) for the vector potential in the situation in which the distribution of current density is confined to a subset $\hat{V} \subset V$ =allspace. Chose the origin near to it. Then a long calculation based on Taylor's theorem yields the following leading large r approximation to $\mathbf{A}(\mathbf{r})$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{m}\wedge\mathbf{r}, \tag{48}$$

where the magnetic dipole moment of the current distribution is defined by

$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\wedge\mathbf{J}(\mathbf{r})d\tau. \tag{49}$$

We consider in detail only the following case.

3.5 The current loop

Here we look at the vector potential $\mathbf{A}$ (33) of a current loop, *i.e.* a wire of negligible cross-section shaped in the form of a closed contour C, carrying a current I. For simplicity, let C define a plane loop of area $\mathbf{S} = S\mathbf{n}$ of unit normal $\mathbf{n}$.

Chose an origin near the loop and seek the vector potential of its magnetic field, at distances large on a scale set by the physical dimensions of the loop. (Or, consider $\mathbf{A}(\mathbf{r})$ due to a small loop.)

Let S be a surface such that $\partial S = C$. Let $\mathbf{c}$ be an arbitrary constant vector, and work on $\mathbf{c}\cdot\mathbf{A}$

$$\begin{aligned}
\mathbf{c}\cdot\mathbf{A} &= \frac{\mu_0 I}{4\pi}\oint_C \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{c}\cdot d\mathbf{r}' \\
&= \frac{\mu_0 I}{4\pi}\int_S \mathbf{n}'\cdot\boldsymbol{\nabla}'\wedge\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{c}\right) dS' \quad \text{(Stokes)} \\
&= \frac{\mu_0 I}{4\pi}\left[\int_S d\mathbf{S}'\wedge\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]\cdot\mathbf{c}.
\end{aligned} \tag{50}$$

Hence

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_S d\mathbf{S}'\wedge\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_S d\mathbf{S}'\wedge\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right). \tag{51}$$

Here, in order to get the leading large r approximation, to $\mathbf{A}(\mathbf{r})$, we simply drop all dependence on $\mathbf{r}'$ from the integrand of (51). So we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[I\int_S d\mathbf{S}'\right]\wedge\mathbf{r} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{m}\wedge\mathbf{r}, \tag{52}$$

where we have defined the magnetic dipole moment of the current loop by

$$\mathbf{m} = I \int_S \mathbf{dS} = IS\mathbf{n}. \tag{53}$$

We have reached a result which agrees exactly with (48). To see this recall the usual conversion $\int_V (\ldots)\mathbf{J}(\mathbf{r})\, d\tau \to \int_C (\ldots) I\mathbf{dr}$. Then (49) gives

$$\mathbf{m} = \frac{1}{2} I \int_C \mathbf{r}\wedge\mathbf{dr} = IS\mathbf{n}, \tag{54}$$

upon use of result of example 6 of the vector calculus revision sheet.

We have obtained a result crucial to the understanding of magnetism at all levels: a small current loop gives, via (54), a physical realisation of a magnetic moment.

3.6 Dipole view of m

We now show why, in the previous section, we referred to **m** as a magnetic dipole moment.

At points where there is no charge density $\mathbf{J} = 0$, the magnetic field **B** obeys

$$\boldsymbol{\nabla}\wedge\mathbf{B} = 0. \tag{55}$$

At such points, we can introduce a magnetic scalar potential Ω via

$$\mathbf{B} = -\boldsymbol{\nabla}\Omega. \tag{56}$$

As $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we have, as in electrostatics,

$$\nabla^2\Omega = 0, \tag{57}$$

Laplace's equation, of which we know various solutions. The one of relevance here is the analogue of the one for the potential of the electric dipole of moment **p** given as (58) of Sec. 2.1, namely

$$\Omega = -\frac{\mu_0}{4\pi}\mathbf{m}\cdot\boldsymbol{\nabla}\frac{1}{r}. \tag{58}$$

¿From this, we can calculate **B** using (56), and caste the result into the form $\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}$, where **A** is given by (52).

[Given the vector potential (52), we calculate the magnetic field **B**. First evaluate

$$\boldsymbol{\nabla}\wedge(\frac{\mathbf{m}\wedge\mathbf{r}}{r^3}) = -\boldsymbol{\nabla}\wedge(\mathbf{m}\wedge\boldsymbol{\nabla}\frac{1}{r}) = -\boldsymbol{\nabla}\wedge(\mathbf{m}\wedge\mathbf{v}), \tag{59}$$

with a temporary abbreviation $\mathbf{v} = \boldsymbol{\nabla}\frac{1}{r}$. Second

$$\begin{aligned}
[\boldsymbol{\nabla}\wedge(\mathbf{m}\wedge\mathbf{v})]_k &= \epsilon_{kij}\partial_i\epsilon_{jpq}m_p v_q = (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip})\partial_i m_p v_q \\
&= \partial_i m_k v_i - \partial_i m_i v_k = m_k\boldsymbol{\nabla}\cdot\mathbf{v} - \mathbf{m}\cdot\boldsymbol{\nabla}v_k \\
&= [\mathbf{m}\,\boldsymbol{\nabla}^2\tfrac{1}{r} - (\mathbf{m}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\tfrac{1}{r}]_k.
\end{aligned} \tag{60}$$

Since we are dealing with non-zero (actually large) r, we can certainly use $\boldsymbol{\nabla}^2\frac{1}{r} = 0$, so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}(\mathbf{m}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\frac{1}{r} = \frac{\mu_0}{4\pi}\boldsymbol{\nabla}(\mathbf{m}\cdot\boldsymbol{\nabla})\frac{1}{r} = -\boldsymbol{\nabla}\left[-\frac{\mu_0}{4\pi}(\mathbf{m}\cdot\boldsymbol{\nabla})\frac{1}{r}\right] = -\boldsymbol{\nabla}\Omega, \tag{61}$$

where Ω is the scalar potential (56).]

We have found a certain analogy between the magnetic dipole moment **m** that determines the leading large r behaviour of the vector potential $\mathbf{A}(\mathbf{r})$ of a current distribution localised near the origin of space, and the electric dipole. Since the 'dipole term' gives the leading contribution to $\mathbf{A}(\mathbf{r})$, this underlines the fact that magnetism has no analogue of the point charge: as far as is known at present magnetic monopoles do not exist. But the small current loop provides a physical realisation of a magnetic dipole.

[**A brief informal aside**

If one considers atoms which possess spin about some axis, one can see roughly that the motion of their electrons approximate to current loops with moments parallel to this axis. If the spin axes of all the atoms, in some material made up of such atoms, can be made to line up parallel, then the material acquires a macroscopic magnetic moment. This offers a little insight into the origin of permanent or (ferro-)magnetism.]

3.7 Forces and couples

¿From (17) of Sec. 1.3, we find that the force, felt by an element of volume δV of medium in which the current density is $\mathbf{J}(\mathbf{r})$, because of a given magnetic field $\mathbf{B}(\mathbf{r})$ is

$$\begin{aligned} \delta\mathbf{F}(\mathbf{r}) &= [\mathbf{J}(\mathbf{r})\delta V]\wedge\mathbf{B}(\mathbf{r}) \quad \text{or} \\ &= I\delta\mathbf{r}\wedge\mathbf{B}(\mathbf{r}). \end{aligned} \tag{62}$$

for an element $\delta\mathbf{r}$ of thin conducting wire carrying current I.

For a loop C_1, carrying current I_1, in a given field $\mathbf{B}$, the total force and couple felt are

$$\mathbf{F} = \oint_{C_1} I_1\mathbf{dr}_1\wedge\mathbf{B}(\mathbf{r}_1) \tag{63}$$

$$\mathbf{G} = \oint_{C_1} \mathbf{r}_1\wedge\left[I_1\mathbf{dr}_1\wedge\mathbf{B}(\mathbf{r}_1)\right]. \tag{64}$$

It can be shown (see problem set 2) that, for C_1 a current loop of moment $\mathbf{m} = IS\mathbf{n}$ in a uniform magnetic field,

$$\mathbf{F} = 0, \quad \mathbf{G} = \mathbf{m}\wedge\mathbf{B}. \tag{65}$$

If $\mathbf{B}_2(\mathbf{r})$ is the field due to a current loop C_2 carrying current I_2

$$\mathbf{B}_2(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint_{C_2}\frac{I_2\mathbf{dr}_2\wedge(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_2|^3}, \tag{66}$$

then the force $\mathbf{F}_{12}$, exerted on loop C_1 by (the magnetic field due to the current in) the loop C_2, is

$$\mathbf{F}_{12} = \oint_{C_1} I_1\mathbf{dr}_1\wedge\mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0}{4\pi}I_1I_2\oint_{C_1}\oint_{C_2}\mathbf{dr}_1\wedge(\mathbf{dr}_2\wedge\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}). \tag{67}$$

It is not obvious, but true, that (67) is compatible with Newton's third law. Proof, which requires the application of Stokes's theorem, is asked for in Problem set 2.

Example: parallel wires

Suppose $C_{1,2}$ are infinite wires carrying currents $I_{1,2}$, the former along the x-axis, the latter parallel to it and through $(0,0,a)$. Use Cartesian coordinates.

Consider the element $I_1\mathbf{dr}_1 = I_1\,dx\mathbf{i}$ at the origin. The force exerted on it by C_2 is

$$\begin{aligned} \mathbf{dF}_1 &= I_1\,dx\mathbf{i}\wedge\mathbf{B}_2(0), \quad \mathbf{B}_2(0) = \frac{\mu_0 I_2}{2\pi a}\,\mathbf{j} \\ &= \frac{\mu_0}{2\pi a} I_1 I_2 \mathbf{k} dx. \end{aligned} \tag{68}$$

This uses the result (38) derived in example a) of Sec. 3.3. It follows that the force per unit length felt by C_1 due to C_2 is

$$\mathbf{F} = \frac{\mu_0}{2\pi a} I_1 I_2 \mathbf{k}. \tag{69}$$

This is a force of attraction for I_1, I_2 of the same sign.

3.8 Proof of (42)

Let S be a surface with unit normal $\mathbf{n}$ which separates regions $V_\pm$ of space, with $\mathbf{n}$ pointing from S into V_+. Let $\mathbf{B}_\pm$ be the magnetic fields in $V_\pm$, and let current $\mathbf{s}$ per unit length flow in S.

Consider an area A of S sufficiently small to be considered plane. Apply Ampére's law

$$\int_C \mathbf{B}.\mathbf{dr} = \mu_0 I \tag{70}$$

to the plane needle shaped contour C shown, in the limit $\delta \to 0$ for finite (small) l. l should be small enough for variation of $\mathbf{B}_\pm$ and $\mathbf{s}$ across it, but non-zero. Also the plane of the needle C contains $\mathbf{n}$, but its orientation, *i.e.* the direction of the normal $\mathbf{b}$, $|\mathbf{b}| = 1$ to the plane, is arbitrary. The direction $\mathbf{t}$, $|\mathbf{t}| = 1$ of the needle is then chosen so that $\mathbf{b}, \mathbf{t}, \mathbf{n}$ form an orthonormal right-handed triad. We get

$$\begin{aligned} (-\mathbf{B}_+ + \mathbf{B}_-).\mathbf{t}l &= \mu_0 \mathbf{s}.\mathbf{b}l \\ (-\mathbf{B}_+ + \mathbf{B}_-).\mathbf{n}\wedge\mathbf{b} &= \mu_0 \mathbf{s}.\mathbf{b} \\ \left[(-\mathbf{B}_+ + \mathbf{B}_-)\wedge\mathbf{n} - \mu_0\mathbf{s}\right].\mathbf{b} &= 0. \end{aligned} \tag{71}$$

Since $\mathbf{b}$ was taken to be in an arbitrary direction in S it follows that

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0 \mathbf{s}. \tag{72}$$