Mathematical Tripos 2004, Part IB : Electromagnetism, Lent 2004
Lecture notes by A.J.Macfarlane, DAMTP

Corrections or comments by email to A.J.Macfarlane@damtp.cam.ac.uk

5 Maxwell's equations

5.1 A historical paradox

In magnetostatics, the equation

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, \tag{1}$$

implies $\nabla \cdot \mathbf{J} = 0$. As $\rho = 0$ in magnetostatics, this is compatible with the continuity equation $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$. However naive application of the integral form of (1)

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS}, \tag{2}$$

to the following situation produced a contradiction, one that Maxwell resolved by generalising (1). The 'capacitor' paradox arises by applying (2) to the two surfaces S_1 and S_2 that are bounded by the same curve C. There is a unique answer for the left-side of (2), but the right-side gives different answers $\mu_0 I$ for S_1 and 0 for S_2.

Maxwell proposed that (1) be changed by addition to a term that made it compatible with $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$. This gives rise (in free space or the vacuum) to

$$\nabla \wedge \mathbf{B} = \mu_0 (\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}), \tag{3}$$

as was shown in Sec. 1.4 to be sufficient to achieve consistency.

How does the use of (3) provide resolution of the paradox? There is an electric field only between the plates. So for S_1, lying outside the plates, we have

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 \int_{S_1} \mathbf{J} \cdot \mathbf{dS} = \mu_0 I. \tag{4}$$

Between the plates, where $\mathbf{J} = 0$, we shall *assume* that $\mathbf{E}$ is uniform so that $\mathbf{E} = \dfrac{\sigma}{\epsilon_0}\mathbf{k}$. Hence

$$\begin{aligned} \frac{1}{\mu_0} \oint_C \mathbf{B} \cdot \mathbf{dr} &= \int_{S_2} \mathbf{J} \cdot \mathbf{dS} + \epsilon_0 \int_{S_2} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS} \\ &= 0 + \epsilon_0 \frac{d}{dt} \int_{S_2} \mathbf{E} \cdot \mathbf{dS} \\ &= \frac{d}{dt}(\sigma A) = \frac{dQ}{dt} = I, \end{aligned} \tag{5}$$

as required for consistency. Here σ is the charge density and A is the plate area. The assumption that $\mathbf{E}$ is uniform is a crude one. It can be avoided by doing a somewhat harder calculation along lines similar to those followed above.

5.2 Energy and energy transport

Recall the field energy formulas

$$W_{el} = \frac{\epsilon_0}{2}\int_V \mathbf{E}^2 d\tau, \quad W_{mag} = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau, \tag{6}$$

and the expression for the rate of Ohmic heat loss *i.e.* the rate of dissipation of electromagnetic energy as heat

$$\int \mathbf{J}\cdot\mathbf{E}d\tau. \tag{7}$$

The Maxwell equation (3) implies

$$\frac{1}{\mu_0}\mathbf{E}\cdot\nabla\wedge\mathbf{B} = \mathbf{E}\cdot\mathbf{J} + \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}. \tag{8}$$

Now

$$\mathbf{E}\cdot\nabla\wedge\mathbf{B} = -\nabla\cdot(\mathbf{E}\wedge\mathbf{B}) + \mathbf{B}\cdot\nabla\wedge\mathbf{E} \quad = -\nabla\cdot(\mathbf{E}\wedge\mathbf{B}) - \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}. \tag{9}$$

Hence

$$\begin{aligned} -\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} &= \mathbf{J}\cdot\mathbf{E} + \frac{1}{\mu_0}\nabla\cdot\mathbf{E}\wedge\mathbf{B} \\ -\frac{d}{dt}\left[\frac{\epsilon_0}{2}\int_V \mathbf{E}^2 d\tau + \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau\right] &= \int_V \mathbf{J}\cdot\mathbf{E}d\tau + \frac{1}{\mu_0}\int_S \mathbf{n}\cdot\mathbf{E}\wedge\mathbf{B}dS. \end{aligned} \tag{10}$$

For the last term the divergence theorem has been applied to a fixed volume V of space bounded by a surface S. The left side here is the rate of decrease of the total field energy $W = W_{el}+W_{mag}$. The first term on the right side of (10) represents the rate of loss of energy as Ohmic heat, while the second term there is the rate of energy transport out of V through the surface S.

For the latter, define the Poynting vector $\mathbf{S}$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\wedge\mathbf{B}. \tag{11}$$

The flux of $\mathbf{S}$ through a closed surface S, with outward unit normal $\mathbf{n}$, is

$$\int_S \mathbf{S}\cdot\mathbf{n}dS. \tag{12}$$

This is the flux of electromagnetic energy being transported through S out of V.

Eq. (10) thus gives a generally applicable account of energy changes in a conducting medium.

5.3 Decay of charge density in a medium of high conductivity σ

In Sec. 1.4, we derived the continuity equation

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \tag{13}$$

from Maxwell's equations. In a conducting medium of conductivity σ we have $\mathbf{J} = \sigma\mathbf{E}$ and hence

$$\nabla\cdot\mathbf{J} = \sigma\nabla\cdot\mathbf{E} = \frac{\sigma}{\epsilon_0}\rho.$$

Now (13) implies

$$\frac{\sigma}{\epsilon_0}\rho + \frac{\partial\rho}{\partial t} = 0, \quad \text{and hence} \quad \rho(t) = \rho(0)\exp(-\frac{t}{\tau}), \tag{14}$$

where $\tau = \frac{\epsilon_0}{\sigma}$ is the *relaxation time* of the medium. For copper or silver $\tau \approx 10^{-18} sec.$, so that any charge density present – for whatever reason – in the medium at the initial time $t = 0$ quickly goes to zero. It may be expected to flow to the surface of the medium. For a perfect conductor, for which σ is infinite, we have $\rho(t) = 0$ at all times, as has been discussed above.

5.4 Plane wave solutions of Maxwell's equations

We here deal with the vacuum or free-space, *i.e.* $\rho = 0, \quad \mathbf{J} = 0$. We begin as simply as possible by seeking a solution describing a wave propagating in the z-direction with fields that do not depend on x or y.

Looking at $\nabla \cdot \mathbf{E} = 0$, we find that E_z is constant. Looking for linearly polarised solutions of wave type, we put $E_z = 0$, and *assume* we can, for all t, chose axes so that

$$\mathbf{E} = (E, 0, 0). \tag{15}$$

Sec. 1.6 proves that the components of $\mathbf{E}$ each satisfy a wave equation. Hence

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}. \tag{16}$$

The solution of such a wave equation can be written as

$$E(z,t) = f(z - ct) + g(z + ct). \tag{17}$$

The f and g terms here describe waves moving respectively in the positive and negative z-directions with speed c. In particular, we can consider a monochromatic wave, one with a fixed angular frequency ω, in which

$$E = E_0 \, \exp i\omega(\frac{z}{c} - t) = E_0 \, \exp i(kz - \omega t) \tag{18}$$

where we have defined the wave-number k by

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \quad . \tag{19}$$

Here $\nu\lambda = \frac{\omega}{2\pi}\lambda = c$ relates the wavelength λ and frequency of the wave in a standard way to other wave variables. Finally, note that the use of complex exponentials is very convenient, but the physical fields must always be identified by taking real parts.

What about the magnetic fields? Looking at $\nabla \cdot \mathbf{B} = 0$, we find that B_z is constant, and take it to be zero. It is natural to assume that $\mathbf{B}$ is of the form

$$\mathbf{B} = \mathbf{B}_0 \exp i(kz - \omega t). \tag{20}$$

Then in $\nabla \wedge \mathbf{E}$ the only no-zero entry is $\frac{\partial E_x}{\partial z}$ so that we have $\mathbf{B}_0 = (0, B_0, 0)$, and hence, from

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{21}$$

we get

$$ikE_0 - i\omega B_0 = 0, \quad B_0 = \frac{E_0}{c}. \tag{22}$$

So our wave solution of Maxwell's equations is

$$\mathbf{E} = (E_0, 0, 0) \, \exp i(kz - \omega t), \quad \mathbf{B} = \frac{1}{c}(0, E_0, 0) \, \exp i(kz - \omega t). \tag{23}$$

It should be checked that (23) satisfies also (the zero current density version of) the Maxwell equation (3), although our use of the fact that each component of $\mathbf{E}$ satisfies a wave equation guarantees it. Thus the simplifying assumptions we have made have led us to the valid and simple wave solution (23) of Maxwell's equations. We could similarly have adopted a choice of axes such that that $\mathbf{E} = (0, E, 0)$, and reached, as above, the solution

$$\mathbf{E} = (0, E_0, 0) \, \exp i(kz - \omega t), \quad \mathbf{B} = (-\frac{1}{c}E_0, 0, 0) \, \exp i(kz - \omega t). \tag{24}$$

The solutions (23) and (24) are linearly independent, and the general monochromatic wave of frequency ω is obtained as a linear superposition of them, has fields $\mathbf{E}$ and $\mathbf{B}$ that are transverse to the direction of propagation of the wave. Also $\mathbf{E} \cdot \mathbf{B} = 0$.

The solutions (23) and (24) are said to be **linearly polarised**, with **polarisation vectors** $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$, giving the directions, for all t, of their electric fields.

To discuss the transport of energy by the wave (23) obtained above, we require the real parts

$$\mathbf{E} = (E_0, 0, 0)\ \cos(kz - \omega t), \quad \mathbf{B} = (0, \frac{1}{c}E_0, 0)\ \cos(kz - \omega t), \tag{25}$$

so that, using (11), we get

$$\mathbf{S} = \frac{1}{\mu_0}\frac{{E_0}^2}{c}\cos^2(kz - \omega t)\,(0, 0, 1). \tag{26}$$

Thus the rate of energy transport across unit area normal to the direction of propagation of the wave (say at $z = 0$) is

$$|\mathbf{S}| = \frac{1}{\mu_0}\frac{{E_0}^2}{c}\cos^2 \omega t. \tag{27}$$

Averaging over one period, $T = \dfrac{2\pi}{\omega}$, of the wave motion, we get for the average rate of energy transport across unit area

$$\langle|\mathbf{S}|\rangle = \frac{\int_0^T |\mathbf{S}|(t)dt}{\int_0^T dt} = \frac{1}{2\mu_0}\frac{{E_0}^2}{c} = \frac{1}{2}\epsilon_0 c E_0^2, \tag{28}$$

since $\epsilon_0\mu_0 = c^{-2}$. The energy density w of the wave (25) can be calculated using (105) of Chapter 2 for w_{el} and (48) of Chapter 4 for w_{mag}. Thus

$$w = w_{el} + w_{mag} = \frac{1}{2}(\epsilon_0 + \frac{1}{\mu_0 c^2})E_0^2\cos^2(kz - \omega t) = \epsilon_0 E_0^2 \cos^2(kz - \omega t). \tag{29}$$

For the time average of this we have

$$\langle w\rangle = \frac{1}{2}\epsilon_0 E_0^2, \tag{30}$$

and hence

$$\langle|\mathbf{S}|\rangle = c\langle w\rangle. \tag{31}$$

For the simple plane wave (25), it follows that the energy density travels at the speed of light across unit area normal to the wave.

Of course, similar results holds for the wave (24).

If we consider a linearly polarised wave with fields

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0\ \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t), \quad \mathbf{B}(\mathbf{r}) = \mathbf{B}_0\ \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t), \tag{32}$$

where $\mathbf{k}$ the wave-vector, with $|\mathbf{k}| = k$, gives the direction of propagation of the wave, (*i.e.* here $\mathbf{k} \neq \mathbf{e}_z$ and the wave number $k \neq 1$). Then $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ implies $\mathbf{E}_0 \cdot \mathbf{k} = 0$, and likewise $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ implies $\mathbf{B}_0 \cdot \mathbf{k} = 0$, so that both these fields are transverse to the direction of propagation. Also (21) implies

$$i\mathbf{k}\wedge\mathbf{E}_0 - i\omega\mathbf{B}_0 = 0, \tag{33}$$

which gives $\mathbf{B}_0$ in terms of $\mathbf{E}_0$. Further the remaining Maxwell equation $\boldsymbol{\nabla}\wedge\mathbf{B} = \dfrac{1}{c^2}\dfrac{\partial\mathbf{E}}{\partial t}$ implies

$$i\mathbf{k}\wedge\mathbf{B}_0 = -i\frac{\omega}{c^2}\mathbf{E}_0, \tag{34}$$

compatibly with (33) iff

$$k^2 = \frac{\omega^2}{c^2}, \quad \text{giving} \quad k = \frac{\omega}{c}. \tag{35}$$

We have merely reproduced our wave in an arbitrary Cartesian basis.

[**Circularly polarised waves**

Take a solution that is (23) minus i-times-(24), with E_0 real. This has physical fields

$$\mathbf{E} = \text{Re}(E_0, -iE_0, 0)\, \exp i(kz - \omega t), \quad \mathbf{B} = \text{Re}\frac{1}{c}(iE_0, E_0, 0)\, \exp i(kz - \omega t) \quad \text{or} \tag{36}$$

$$\mathbf{E} = E_0(\cos(kz - \omega t), \sin(kz - \omega t), 0) \quad , \quad \mathbf{B} = \frac{E_0}{c}(-\sin(kz - \omega t), \cos(kz - \omega t), 0) \quad \text{or} \tag{37}$$

$$\mathbf{E} = E_0 \mathbf{e}_s(kz - \omega t) \quad , \quad \mathbf{B} = \frac{E_0}{c}\mathbf{e}_\phi(kz - \omega t, \tag{38}$$

where $\mathbf{e}_s(\phi)$ and $\mathbf{e}_\phi(\phi)$ are the unit vectors of cylindrical polar coordinates (s, ϕ, z) with the z-axis in the direction of propagation of the wave. The wave (38) is said to be (positively) circularly polarised. A wave of negative circular polarisation linearly independent of this can be constructed, using (23) plus i-times-(24) with E_0 real, but we do not need the details contained in this parenthesis].

5.5 Boundary conditions

It seems there is going to be time to cover this in lectures. Sec. 1.7 should perhaps be reviewed at this point.

Suppose a surface S carries either a charge density σ per unit area, or a surface current s per unit length. Let the unit normal $\mathbf{n}$ to S point from the negative $(-)$ to the positive $(+)$ side of S.

We proved in Sec. 2.2, the discontinuity formula

$$\mathbf{n}.\mathbf{E}|_-^+ = \frac{1}{\epsilon_0}\sigma, \tag{39}$$

and in Sec. 3.8, that

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0 \mathbf{s}. \tag{40}$$

It should be clear that the proofs can be applied to deriving

$$\mathbf{n}.\mathbf{B}|_-^+ = 0, \tag{41}$$

and

$$\mathbf{n}\wedge\mathbf{E}|_-^+ = 0. \tag{42}$$

As an aid to remembering these results, we noted in Sec. 1.7, their exact correspondence with Maxwell's equations themselves.

Note that $\mathbf{n}.\mathbf{v}$ and $\mathbf{n}\wedge\mathbf{v}$ give the **normal** and **tangential** components of any vector $\mathbf{v}$. It is obvious that the tangential component satisfies $\mathbf{n}.(\mathbf{n}\wedge\mathbf{v}) = 0$.

Consider then a perfect conductor $\mathcal{C}$ with surface S and normal $\mathbf{n}$ pointing into the conducting medium, in which $\mathbf{E} = 0$ and $\mathbf{B} = 0$. Then the boundary conditions just inside the free space (negative) side of S demand the vanishing of the normal component of $\mathbf{B}$ and of the tangential component of $\mathbf{E}$. This follows (41,42). Eqs. (39,40) are usually subsequently used to *calculate* σ and s for S.

5.6 Reflection at the surface of a perfect conductor

We consider a monochromatic wave (23) propagating in the z-direction from the half-space $z < 0$, towards perfectly conducting material in $z > 0$, whose surface is the plane $z = 0$. In fact the solution of Maxwell's equations plus the boundary conditions (BC) on $z = 0$ will comprise not only an incident wave but also (at least) a suitably matched reflected wave. The fields of the former will have argument $(kz - \omega t)$, where $kc = \omega$, while those of the latter (moving in the negative z-direction) are $(-kz - \omega t)$. All fields in the problem have the same t-dependence $\propto e^{-i\omega t}$.

We know that the fields $\mathbf{E}$ and $\mathbf{B}$ are zero inside perfectly conducting media, it therefore follows the BC are: tangential $\mathbf{E}$ and normal $\mathbf{B}$ are zero at $z = 0$. For the wave (23) this just means that $E_x = 0$ at $z = 0$. Thus for the electric fields of the incident and reflected parts of our total wave solution of Maxwell's equations, we take

$$\mathbf{E}_{inc} = (E_0, 0, 0)\,\exp i(kz - \omega t), \quad \mathbf{E}_{ref} = (-E_0, 0, 0)\,\exp i(-kz - \omega t), \tag{43}$$

since their superposition

$$\mathbf{E} = \mathbf{E}_{inc} + \mathbf{E}_{ref}, \tag{44}$$

by construction, gives $E_x = 0$ at $z = 0$. The corresponding magnetic fields are $\mathbf{B} = \mathbf{B}_{inc} + \mathbf{B}_{ref}$ with

$$\mathbf{B}_{inc} = \frac{1}{c}(0, E_0, 0)\,\exp i(kz - \omega t), \quad \mathbf{B}_{ref} = \frac{1}{c}(0, E_0, 0)\,\exp i(-kz - \omega t). \tag{45}$$

We see from this that $\mathbf{B}$ does have a non-zero tangential component at $z = 0$, namely

$$\mathbf{B} = 2\frac{1}{c}(0, E_0, 0)\,e^{-i\omega t}. \tag{46}$$

But this just tells us that a surface current $\mathbf{s}$ necessarily accompanies the fields $\mathbf{E}$ and $\mathbf{B}$ in a consistent solution of Maxwell's equations and boundary conditions.

Recalling the formula (42) of chapter three for $\mathbf{s}$

$$\mathbf{n}\wedge\mathbf{B}|_{-}^{+} = \mu_0\mathbf{s}, \tag{47}$$

we obtain

$$\mu_0\mathbf{s} = -\,\mathbf{n}\wedge\mathbf{B}|_{-} = \frac{2E_0}{c}e^{-i\omega t}\,(1, 0, 0). \tag{48}$$

5.7 The historical paradox revisited

We return to the topic of Sec. 5.1, to provide a treatment which does not make the (crude) assumption that the the electric field $\mathbf{E}$ beteween the plates is uniform. Assume the plates are circular of radius a, and neglect edge effects. Use cylindrical polars (s, ϕ, z).

We shall treat the case in which

$$\mathbf{E} = E_z(s)\mathbf{k}\exp(-i\omega t), \quad \mathbf{B} = B_\phi(s)\mathbf{e}_\phi\exp(-i\omega t). \tag{49}$$

The Maxwell equation $\nabla\wedge\mathbf{E} + \dfrac{\partial\mathbf{B}}{\partial t} = 0$ has only got a non-trivial $\mathbf{e}_\phi$ component, which gives

$$-\frac{\partial E_z}{\partial s} + (-i\omega)B_\phi = 0. \tag{50}$$

The Maxwell equation

$$\nabla\wedge\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{51}$$

between the plates, where $\mathbf{J} = 0$, has only got a non-trivial z component

$$\frac{1}{s}\frac{\partial}{\partial s}(sB_\phi) = -i\frac{\omega}{c^2}E_z \quad \text{using} \quad \epsilon_0\mu_0 = c^{-2}. \tag{52}$$

Substituting for B_ϕ from (50) into (52), we find

$$\frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial E_z}{\partial s}) + \frac{\omega^2}{c^2}E_z = 0. \tag{53}$$

We set $k = \dfrac{\omega}{c}$, and recognize (53) as the equation satisfied by the Bessel function $J_0(ks)$. Hence, we write

$$E_z = \alpha J_0(ks), \quad B_\phi = i\frac{1}{\omega}\frac{\partial E_z}{\partial s} = i\frac{\alpha}{\omega}\frac{\partial J_0(ks)}{\partial s}, \tag{54}$$

where α is a constant.

The surface charge density on the the lower plate is

$$\sigma = \epsilon_0 \mathbf{k}\cdot\mathbf{E}|_-^+ = \epsilon_0 \alpha J_0(ks)\exp(-i\omega t), \quad 0 \le s \le a. \tag{55}$$

We now show that the integral form of (51) can be applied consistently to $\oint_C \mathbf{B}\cdot\mathbf{dr}$ whether or not the surface $S, \partial C = S$, chosen passes between the plates or not. Let C be the circumference of the lower plate, S_2 the lower plate itself, and S_1 a surface bounded by C but lying entirely outside the region between the plates and so pierced by the current I. As before, for S_1 $\oint_C \mathbf{B}\cdot\mathbf{dr} = \mu_0 I$. For S_2, on the other hand, we have

$$\begin{aligned}
\mu_0 I &= \mu_0\frac{dQ}{dt} = \mu_0\frac{d}{dt}\int_{S_2}\sigma dS \\
&= 2\pi\mu_0\frac{d}{dt}\int_0^a s\sigma ds \\
&= 2\pi\mu_0(-i\omega)\exp(-i\omega t)\int_0^a s\epsilon_0\alpha J_0(ks)ds \\
&= -2\pi i\frac{1}{\omega}\frac{\omega^2}{c^2}\exp(-i\omega t)\int_0^a s\alpha J_0(ks)ds \\
&= 2\pi i\frac{\alpha}{\omega}\exp(-i\omega t)\int_0^a(-k^2 sJ_0(ks))ds \\
&= 2\pi i\frac{\alpha}{\omega}\exp(-i\omega t)\int_0^a\frac{\partial}{\partial s}(s\frac{\partial J_0(ks)}{\partial s})ds \\
&= 2\pi i\frac{\alpha}{\omega}\exp(-i\omega t)a\frac{\partial J_0(ks)}{\partial s}\Big|_{s=a} = 2\pi a B_\phi(a)\exp(-i\omega t) = \oint_C \mathbf{B}\cdot\mathbf{dr},
\end{aligned} \tag{56}$$

as required. The third line here uses (55), the fourth $\epsilon_0\mu_0 = c^{-2}$, the fifth $k = \omega/c$, the sixth Bessel's equation, the seventh (54) for B_ϕ.

5.8 Addition to Sec. 5.6

¿From (43) and (44), we see that the physical field is the real part of $\mathbf{E}$, *i.e.* for E_0 real

$$(\mathbf{E}_{phys})_x = 2E_0 \sin kz\, \sin\omega t, \tag{57}$$

and similarly

$$(\mathbf{B}_{phys})_x = (2/c)E_0 \cos kz\, \cos\omega t. \tag{58}$$

Since the magnitude $|\mathbf{S}|$ of the Poynting vector is proportional to $\sin\omega t\, \cos\omega t$, its mean value over one period of the wave motion is zero (which makes good sense?). (48) implies that the physical surface current $\mathbf{s}$ is given by

$$\mu_0 \mathbf{s} = (2/c)\cos\omega t. \tag{59}$$

We may now use (52) of Sec. 1.7, to calculate the force $\mathbf{f}$ per unit area exerted on the surface $z = 0$ of the conducting medium. It is

$$\mathbf{f} = \frac{1}{2} s_y (\mathbf{B}_{phys})_y = \frac{1}{2\mu_0}\,\frac{4}{c^2} {E_0}^2 \cos^2\omega t. \tag{60}$$

Hence the mean force per unit area is

$$\langle f \rangle = \epsilon_0 {E_0}^2, \tag{61}$$

using the result $\langle \cos^2\omega t\rangle = \frac{1}{2}$. Since the force is normal to the surface, (61) gives the mean pressure (**radiation pressure**) at the surface.

5.9 Proof of the result $\mathbf{G} = \mathbf{m}\wedge\mathbf{B}$

Refer to Sec. 3.7, **Force and couples**, and supply the proof that the couple exerted by a uniform magnetic field $\mathbf{B}$ on a plane current loop, of area A, unit normal $\mathbf{n}$, carrying current I, is given by (65) there, *i.e.*

$$\mathbf{G} = \mathbf{m}\wedge\mathbf{B}, \quad \mathbf{m} = IA\mathbf{n}. \tag{62}$$

This was also quoted as (35) of Sec. 4.4, and used there. Letting $\mathbf{c}$ be an arbitrary constant vector, we have

$$\begin{aligned} \mathbf{c}\cdot\mathbf{G} &= \mathbf{c}\cdot\oint_C \mathbf{r}\wedge(I\mathbf{dr}\wedge\mathbf{B}) = I\oint_C \mathbf{c}\cdot(\mathbf{r}\cdot\mathbf{B}\;\mathbf{dr} - \mathbf{r}\cdot\mathbf{dr}\;\mathbf{B}) \\ &= I\oint_C [\mathbf{c}\cdot(\mathbf{r}\cdot\mathbf{B}\mathbf{dr}) - (\mathbf{c}\cdot\mathbf{B})(\mathbf{r}\cdot\mathbf{dr})]\,. \end{aligned} \tag{63}$$

We now apply Stokes's theorem to each of the terms of (63). For the second term we have

$$\oint_C \mathbf{r}\cdot\mathbf{dr} = \int_S \mathbf{n}\cdot(\boldsymbol{\nabla}\wedge\mathbf{r})dS = 0. \tag{64}$$

For the first term, moving a scalar product in an allowed way, we have

$$I\oint_C (\mathbf{r}\cdot\mathbf{B}\;\mathbf{c})\cdot\mathbf{dr} = I\int_S \mathbf{n}\cdot\boldsymbol{\nabla}\wedge(\mathbf{r}\cdot\mathbf{B}\;\mathbf{c})\,dS = I\int_S \mathbf{n}\cdot\mathbf{B}\wedge\mathbf{c}\,dS = I(\int_S \mathbf{dS})\wedge\mathbf{B}\cdot\mathbf{c}. \tag{65}$$

Here we have used the elementary result $\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{B}) = \mathbf{B}$, for constant $\mathbf{B}$. We may finally detach $\mathbf{c}$ from (65), and get the required result

$$\mathbf{G} = I(\int_S \mathbf{dS})\wedge\mathbf{B} = (IA\mathbf{n})\wedge\mathbf{B} = \mathbf{m}\wedge\mathbf{B}. \tag{66}$$

5.10 List of corrections already inserted into webpage version of the Lecture Notes for IB: Electromagnetism

These have been made without changing the page beginnings and endings of the pages circulated during lectures. They include...

P3: 3 lines below (13) ... which, for qv positive ...
P4: (NB) line below (22) ...flux of $\mathbf{J}$ out of...
P9: last line ... (69) satisfies Poisson's equation for ...
P11: (NB) c) third line now begins: with ends at $z = h$ and $z = -h$...
next line ... for some $E = E(h)$ ...
end of (81) $E = \dfrac{1}{2\epsilon_0}\sigma, \qquad$ indept of h.
P12: f) ... σ at end of first sentence changed to ρ.
P13: line below (91) ... $= -\dfrac{\partial \phi_1}{\partial z}\mathbf{k} = ...$
P13: Original wording in early paragraphs of Sec. 2.3 seriously inadequate. Look at replaced text
P18: (7) ...$\phi = d - c\theta, \quad c,\ d$ constants ...
P19: (26) $\qquad 0 = \psi + \boldsymbol{\nabla}\chi$...
P20: j_k has been replaced by J_k in (30) and (31).
P23: (49) should read $\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\wedge\mathbf{J}(\mathbf{r})d\tau$.
P26: $\mathbf{J}$ has been replaced by $\mathbf{j}$ in (68). Here $\mathbf{j} = (0, 1, 0)$.
P31: same correction as on P26 twice near (23).
P32: wording of later sentences of Sec. 4.4 improved.
P35: (NB) LHS of (10) corrected to read $-\dfrac{d}{dt}\left[\dfrac{\epsilon_0}{2}\displaystyle\int_V \mathbf{E}^2 d\tau + \dfrac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau\right]$
P36: wording before (15) and (24) has been improved.
P37: line below (27) ... period, $T = \dfrac{2\pi}{\omega}$... , and
(30) ... $\langle w\rangle = \frac{1}{2}\epsilon_0 {E_0}^2$.
(34) Correct RHS is $-i\dfrac{\omega}{c^2}\mathbf{E}_0$
P39: RHS (48) ... $\dfrac{2E_0}{c}e^{-i\omega t}(1, 0, 0)$

(NB) Error, not in original, editted carelessly into existing web-page text.
P23: At end of (46) ... $\dfrac{\mu_0}{2}Js$ is correct, *i.e.* J is correct here; I is wrong.

Example Sheet 2.
Q6: no π in denominator of expression for B.
Q7: (i): ... force $\mathbf{F}$ per unit volume, (ii) ... $-\boldsymbol{\nabla}p + \mathbf{F} = 0$ ... , and (iii) $p(s) = \frac{1}{4}\mu_0 J^2(s^2 - a^2)$.