1. A monochromatic plane wave, propagates in empty space \( z < 0 \) with fields

\[
E_{\text{inc}} = e_x \text{Re} \left( \alpha e^{i(kz - \omega t)} \right) \quad B_{\text{inc}} = \frac{1}{c} e_y \text{Re} \left( \alpha e^{i(kz - \omega t)} \right)
\]

A perfect conductor fills the region \( z \geq 0 \). Show that if the reflected fields are given by

\[
E_{\text{ref}} = -e_x \text{Re} \left( \alpha e^{i(-kz - \omega t)} \right) \quad B_{\text{ref}} = \frac{1}{c} e_y \text{Re} \left( \alpha e^{i(-kz - \omega t)} \right)
\]

then the total fields \( E = E_{\text{inc}} + E_{\text{ref}} \) and \( B = B_{\text{inc}} + B_{\text{ref}} \) satisfy the Maxwell equations and the relevant boundary conditions at \( z = 0 \).

What surface current flows in the plane \( z = 0 \)? Compute the Poynting vector in the region \( z < 0 \) and determine its value averaged over a period \( T = 2\pi/\omega \).

Recall from Q5 of sheet 2 that a surface current experiences a Lorentz force from the average magnetic field on either side of the surface. Use this to show that the time-averaged force per unit area on the conductor is \( \langle f \rangle = \varepsilon_0 |\alpha|^2 \).

2. Perfectly conducting plates are positioned at \( y = 0 \) and \( y = a \). Show that a monochromatic plane wave can propagate between the plates in the \( y \) direction only if the frequency is given by \( \omega = n\pi c/a \) with \( n \in \mathbb{Z} \).

3. Perfectly conducting plates are positioned at \( y = 0 \) and \( y = a \). Show that a monochromatic wave may propagate between the plates in the direction \( z \) if the field components are

\[
E_x = \omega A \sin \left( \frac{n\pi y}{a} \right) \sin(kz - \omega t)
\]

and

\[
B_y = kA \sin \left( \frac{n\pi y}{a} \right) \sin(kz - \omega t) \quad B_z = \frac{n\pi A}{a} \cos \left( \frac{n\pi y}{a} \right) \cos(kz - \omega t)
\]

with \( A \) a constant and \( n \in \mathbb{Z} \). Show that the wavelength \( \lambda \) is given by \( 1/\lambda^2 = 1/\lambda_{\infty}^2 - n^2/4a^2 \), where \( \lambda_{\infty} \) is the wavelength of the same frequency in the absence of conducting plates.

4. Consider a plane polarized electromagnetic wave described by the vector and scalar potentials \( A(t, x) = \text{Re} \left( A_0 e^{i(kx - \omega t)} \right) \) and \( \Phi(t, x) = \text{Re} \left( \Phi_0 e^{i(kx - \omega t)} \right) \) with constant \( A_0 \) and \( \Phi_0 \). Use Maxwell’s equations to find a relationship between \( A_0 \) and \( \Phi_0 \).

Find a gauge transformation such that the new vector potential is “transversely polarised”, i.e. \( A_0 \cdot k = 0 \). What is the scalar potential \( \Phi \) in this gauge?

5. (a) A tensor of type \((0, 2)\) has components \( T_{\mu\nu} \). View these components as a \( 4 \times 4 \) matrix. Show that if this matrix is invertible in one inertial frame then it is invertible in any inertial frame, and that the components of the inverse matrix \( (T^{-1})^{\mu\nu} \) define a tensor of type \((2, 0)\).

(b) Show that the object with components \( \varepsilon_{\mu\nu\rho\sigma} \) w.r.t. any inertial frame is an isotropic pseudo-tensor of type \((0, 4)\).
6. A particle of rest mass \( m \) and charge \( q \) moves in a constant uniform electric field \( \mathbf{E} = (E, 0, 0) \). It starts from the origin with initial 3-momentum \( \mathbf{p} = (0, p_0, 0) \). Show that the particle traces out a path in the \((x, y)\) plane given by

\[
x = \frac{\mathcal{E}_0}{qE} \left( \cosh \left( \frac{qEy}{p_0c} \right) - 1 \right)
\]

where \( \mathcal{E}_0 = \sqrt{p_0^2 c^2 + m^2 c^4} \) is the initial kinematic energy of the particle.

7. For constant electric and magnetic fields, \( \mathbf{E} \) and \( \mathbf{B} \), show that if \( \mathbf{E} \cdot \mathbf{B} = 0 \) and \( \mathbf{E}^2 - c^2 \mathbf{B}^2 \neq 0 \) then there exist inertial frames where either \( \mathbf{E} \) or \( \mathbf{B} \) are zero, but not both. [Hint: show that you can choose axes so that only \( E_y \) and \( B_z \) are non-zero and then consider a Lorentz transformation in the \( x \)-direction.]

8. An electromagnetic wave is reflected by a perfect conductor at \( x = 0 \). The electric field is \( \mathbf{E}(t, x) = e_y[f(t_-) - f(t_+)] \) where \( f \) is an arbitrary function and \( ct_\pm = ct \pm x \). Show that this satisfies the relevant boundary condition at the conductor. Find the corresponding magnetic field \( \mathbf{B} \).

Show that under a Lorentz transformation to an inertial frame moving with speed \( v \) in the \( x \)-direction the electric field is transformed to

\[
\mathbf{E}'(t', x') = e_y \left[ \rho f(\rho t') - \frac{1}{\rho} f \left( \frac{t'}{\rho} \right) \right]
\]

where \( \rho = \sqrt{\frac{c-v}{c+v}} \).

Hence for an incident wave \( \mathbf{E}(t, x) = e_y F(t_-) \), find the wave that is reflected after it hits a perfectly conducting mirror moving with speed \( v \) in the \( x \)-direction.

9. (a) A scalar field \( \Phi \) obeys the wave equation \( \partial^\mu \partial_\mu \Phi = 0 \). Its energy-momentum tensor is \( T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi \). Show that \( T_{\mu\nu} \) is conserved: \( \partial_\mu T^{\mu\nu} = 0 \).

(b) The energy-momentum tensor of the Maxwell field is \( T_{\mu\nu} = \mu_0^{-1} \left( F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \).

Explain how \( T_{00} \) and \( T_{0i} \) are related to the energy density and Poynting vector of the electromagnetic field. Show that Maxwell’s equations imply that \( \partial_\nu T^{\mu\nu} = -F^{\mu\nu} j^\nu \) and that the time component of this equation is the energy conservation equation for Maxwell’s theory.

10. (⋆) For a general 4-velocity, written as \( U^\mu = \gamma(c, \mathbf{v}) \), show that

\[
F^{\mu\nu} U_\nu = \gamma \left( \frac{\mathbf{E} \cdot \mathbf{v}}{c} + \frac{\mathbf{E} + \mathbf{v} \times \mathbf{B}}{c} \right)
\]

In the rest-frame of a conducting medium, Ohm’s law states that \( \mathbf{J} = \sigma \mathbf{E} \) where \( \sigma \) is the conductivity and \( \mathbf{J} \) is the 3-current. Assuming that \( \sigma \) is a Lorentz scalar, show that Ohm’s law can be written covariantly as

\[
j^\mu + \frac{1}{c^2} (j^\nu U_\nu) U^\mu = \sigma F^{\mu\nu} U_\nu
\]

where \( j^\mu \) is the charge-current density and \( U^\mu \) is the (uniform) 4-velocity of the medium. If the medium moves with 3-velocity \( \mathbf{v} \) in some inertial frame, show that the current in that frame is

\[
\mathbf{J} = \rho \mathbf{v} + \sigma c \mathbf{v} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{c^2} (\mathbf{v} \cdot \mathbf{E}) \mathbf{v} \right)
\]

where \( \rho \) is the charge density. Simplify this formula, given that the charge density vanishes in the rest-frame of the medium.