Electromagnetism: Example Sheet 3

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1. A steady current I flows along a cylindrical conductor of constant circular cross-section and uniform conductivity σ. Show, using the relevant equations for $\mathbf{E}$ and $\mathbf{J}$, that the current is distributed uniformly across the cross-section of the cylinder, and calculate the electric and magnetic fields just outside the surface of the cylinder.

Verify that the integral of the Poynting vector over unit length of the surface is equal to the rate per unit length of dissipation of electrical energy as heat.

2. A monochromatic wave, with fields $\mathbf{E}_{\text{inc}} = E_0\hat{\mathbf{x}}\, e^{i(kz-\omega t)}$ and $\mathbf{B}_{\text{inc}} = (E_0/c)\hat{\mathbf{y}}\, e^{i(kz-\omega t)}$, propagates in empty space $z < 0$. A perfect conductor fills the region $z \geq 0$. Show that if the reflected fields are given by

$$\mathbf{E}_{\text{ref}} = -E_0\hat{\mathbf{x}}\, e^{i(-kz-\omega t)} \quad , \quad \mathbf{B}_{\text{ref}} = \frac{E_0}{c}\hat{\mathbf{y}}\, e^{i(-kz-\omega t)}$$

then the total fields $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$ and $\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{ref}}$ satisfy the Maxwell equations and the relevant boundary conditions at $z = 0$.

What surface current flows in the plane $z = 0$? Compute the Poynting vector in the region $z < 0$ and compute its value averaged over a period $T = 2\pi/\omega$.

Recall from Q6, Sheet 2, that a surface current experiences a Lorentz force from the average magnetic field on either side of the surface. Use this to show that the time-averaged force per unit area on the conductor is $\bar{f} = \epsilon_0 E_0^2$.

3. Perfectly conducting planes are positioned at $y = 0$ and $y = a$. Show that a monochromatic plane wave can propagate between the plates in the y direction only if the frequency is given by $\omega = n\pi c/a$ with $n \in \mathbf{Z}$.

4. Perfectly conducting planes are positioned at $y = 0$ and $y = a$. Show that a monochromatic wave may propagate between the plates in the direction z if the field components are

$$\begin{aligned} E_x &= \omega A \sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) \\ B_y &= kA\sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) \\ B_z &= \frac{n\pi A}{a}\cos\left(\frac{n\pi y}{a}\right)\cos(kz-\omega t) \end{aligned}$$

with A a constant and $n \in \mathbf{Z}$. Show that the wavelength λ is given by $1/\lambda^2 = 1/\lambda_\infty^2 - n^2/4a^2$, where λ_∞ is the wavelength of waves of the same frequency in the absence of conducting plates.

5. Consider a plane polarized electromagnetic wave described by the vector and scalar potentials,

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad \text{and} \quad \phi(\mathbf{r}, t) = \phi_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

with constant $\mathbf{A}_0$ and ϕ_0. Use Maxwell's equations to find a relationship between $\mathbf{A}_0$ and ϕ_0.

Find a gauge transformation such that the new vector potential is "transversely polarised", i.e. $\mathbf{A}_0 \cdot \mathbf{k} = 0$. What is the scalar potential ϕ in this gauge?

6. For constant electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, show that if $\mathbf{E}\cdot\mathbf{B} = 0$ and $\mathbf{E}^2 - c^2\mathbf{B}^2 \neq 0$ then there exist frames of reference where either $\mathbf{E}$ or $\mathbf{B}$ are zero, but not both.
[Hint: It suffices to take E_y and B_z non zero and consider x-direction Lorentz transfs with $v < c$.]

7. An electromagnetic wave is reflected by a perfect conductor at $x = 0$. The electric field is

$$\mathbf{E}(t, \mathbf{x}) = \widehat{\mathbf{y}}\,[f(t_-) - f(t_+)]$$

where f is an arbitrary function and $ct_\pm = ct \pm x$. Show that this satisfies the relevant boundary condition at the conductor. Find the corresponding magnetic field $\mathbf{B}$.

Show that under a Lorentz transformation to a frame moving with speed v in the x-direction the electric field is transformed to

$$\mathbf{E}'(t', \mathbf{x}') = \widehat{\mathbf{y}}\left[\rho f(\rho t'_-) - \frac{1}{\rho} f\left(\frac{t'_+}{\rho}\right)\right] \quad \text{where} \quad \rho = \sqrt{\frac{c-v}{c+v}}$$

Hence for an incident wave $\mathbf{E}(t, \mathbf{x}) = \widehat{\mathbf{y}} F(t_-)$, find the wave that is reflected after it hits a perfectly conducting mirror moving with speed v in the x-direction.

8. In $d + 1$ space-time dimensions, the equations of electromagnetism are given by

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{where} \quad \mu, \nu = 0, 1, \ldots d$$

How many components does the electric field have? How many components does the magnetic field have? What is the potential energy between two electric charges q_1 and q_2? How many independent, linear polarisations does an electromagnetic wave have?

[Note: Pay particular attention to the special cases $d = 1$ and $d = 2$, partly because they can actually be realised in experiment. For $d \geq 4$, you may denote the area of a $(d-1)$-dim sphere as S_{d-1}.]

9. A particle of rest mass m and charge q moves in a constant uniform electric field $\mathbf{E} = (E, 0, 0)$. It starts from the origin with initial momentum $\mathbf{p} = (0, p_0, 0)$. Show that the particle traces out a path in the (x, y) plane given by

$$x = \frac{\mathcal{E}_0}{qE}\left(\cosh\left(\frac{qEy}{p_0 c}\right) - 1\right)$$

where $\mathcal{E}_0 = \sqrt{p_0^2 c^2 + m^2 c^4}$ is the initial kinematic energy of the particle.

10*. For a general 4-velocity, written as $U_\mu = \gamma(c, -\mathbf{v})$, show that

$$F^{\mu\nu} U_\nu = \gamma \begin{pmatrix} \mathbf{E} \cdot \mathbf{v}/c \\ \mathbf{E} + \mathbf{v} \times \mathbf{B} \end{pmatrix}$$

In the rest-frame of a conducting medium, Ohm's law states that $\mathbf{J} = \sigma \mathbf{E}$ where σ is the conductivity and $\mathbf{J}$ is the 3-current. Assuming that σ is a Lorentz scalar, show that Ohm's law can be written covariantly as

$$J^\mu - \frac{1}{c^2}(J^\nu U_\nu) U^\mu = \sigma F^{\mu\nu} U_\nu$$

where J^μ is the 4-current and U^μ is the (uniform) 4-velocity of the medium. If the medium moves with 3-velocity $\mathbf{v}$ in some inertial frame, show that the current in that frame is

$$\mathbf{J} = \rho \mathbf{v} + \sigma\gamma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{c^2}(\mathbf{v} \cdot \mathbf{E})\mathbf{v}\right)$$

where ρ is the charge density. Simplify this formula, given that the charge density vanishes in the rest-frame of the medium.