Mathematical Tripos Part IB Gordon Ogilvie

ELECTROMAGNETISM Examples 3 Lent Term 2025

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1. A monochromatic plane wave propagates in empty space $z < 0$ with fields

$$\mathbf{E}_{\text{inc}} = \text{Re}\left(\alpha\, e^{i(kz-\omega t)}\right)\mathbf{e}_x\,, \qquad \mathbf{B}_{\text{inc}} = \frac{1}{c}\,\text{Re}\left(\alpha\, e^{i(kz-\omega t)}\right)\mathbf{e}_y\,.$$

A perfect conductor fills the region $z \geq 0$. Show that, if the reflected fields are given by

$$\mathbf{E}_{\text{ref}} = -\text{Re}\left(\alpha\, e^{i(-kz-\omega t)}\right)\mathbf{e}_x\,, \qquad \mathbf{B}_{\text{ref}} = \frac{1}{c}\,\text{Re}\left(\alpha\, e^{i(-kz-\omega t)}\right)\mathbf{e}_y\,,$$

then the total fields $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$ and $\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{ref}}$ satisfy the Maxwell equations and the relevant boundary conditions at $z = 0$.

What surface current flows in the plane $z = 0$? Compute the Poynting vector in the region $z < 0$ and determine its value averaged over a period $T = 2\pi/\omega$.

Recall from Q5 of sheet 2 that a surface current experiences a Lorentz force from the average magnetic field on either side of the surface. Use this to show that the time-averaged force per unit area on the conductor is $\langle f \rangle = \epsilon_0 |\alpha|^2$.

2. Perfectly conducting plates are positioned at $y = 0$ and $y = a$. Show that a monochromatic plane wave can propagate between the plates in the y-direction only if the frequency is given by $\omega = n\pi c/a$ with $n \in \mathbb{Z}$.

3. Perfectly conducting plates are positioned at $y = 0$ and $y = a$. Show that a monochromatic wave can propagate between the plates in the z-direction if the field components are

$$E_x = \omega A \sin\left(\frac{n\pi y}{a}\right)\sin(kz - \omega t)$$

and

$$B_y = kA \sin\left(\frac{n\pi y}{a}\right)\sin(kz - \omega t)\,, \qquad B_z = \frac{n\pi A}{a}\cos\left(\frac{n\pi y}{a}\right)\cos(kz - \omega t)\,,$$

with $A =$ constant and $n \in \mathbb{Z}$. Show that the wavelength λ is given by $1/\lambda^2 = 1/\lambda_\infty^2 - n^2/4a^2$, where λ_∞ is the wavelength of waves of the same frequency in the absence of conducting plates.

4. Consider a plane-polarized electromagnetic wave described by the vector and scalar potentials $\mathbf{A}(\mathbf{x}, t) = \text{Re}\left(\mathbf{A}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$ and $\Phi(\mathbf{x}, t) = \text{Re}\left(\Phi_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$ with constants $\mathbf{A}_0$ and Φ_0. Use Maxwell's equations to find a relationship between $\mathbf{A}_0$ and Φ_0.

Find a gauge transformation such that the new vector potential is 'transversely polarized', i.e. $\mathbf{A}_0 \cdot \mathbf{k} = 0$. What is the scalar potential Φ in this gauge?

5. (a) A tensor of type $(0, 2)$ has components $T_{\mu\nu}$. View these components as a 4×4 matrix. Show that if this matrix is invertible in one inertial frame then it is invertible in any inertial frame, and that the components of the inverse matrix $(T^{-1})^{\mu\nu}$ define a tensor of type $(2, 0)$.

(b) Show that the object with components $\epsilon_{\mu\nu\rho\sigma}$ with respect to any inertial frame is an isotropic pseudo-tensor of type $(0, 4)$.

6. A particle of rest mass m and charge q moves in a constant, uniform electric field $\mathbf{E} = (E, 0, 0)$. It starts from the origin with initial 3-momentum $\mathbf{p} = (0, p_0, 0)$. Show that the particle traces out a path in the (x, y) plane given by

$$x = \frac{\mathcal{E}_0}{qE}\left(\cosh\left(\frac{qEy}{p_0 c}\right) - 1\right),$$

where $\mathcal{E}_0 = \sqrt{p_0^2 c^2 + m^2 c^4}$ is the initial kinematic energy of the particle.

7. For constant electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, show that if $\mathbf{E}\cdot\mathbf{B} = 0$ and $\mathbf{E}^2 - c^2\mathbf{B}^2 \neq 0$ then there exist inertial frames where either $\mathbf{E}$ or $\mathbf{B}$ is zero, but not both. [*Hint:* Show that you can choose axes so that only E_y and B_z are non-zero and then consider a Lorentz transformation in the x-direction.]

8. An electromagnetic wave is reflected by a perfect conductor at $x = 0$. The electric field is $\mathbf{E}(\mathbf{x}, t) = [f(t_-) - f(t_+)]\,\mathbf{e}_y$, where f is an arbitrary function and $ct_\pm = ct \pm x$. Show that this satisfies the relevant boundary condition at the conductor. Find the corresponding magnetic field $\mathbf{B}$.

   Show that under a Lorentz transformation to an inertial frame moving with speed v in the x-direction the electric field is transformed to

$$\mathbf{E}'(\mathbf{x}', t') = \left[\rho f(\rho t'_-) - \frac{1}{\rho} f\left(\frac{t'_+}{\rho}\right)\right]\mathbf{e}_y\,, \qquad \text{where} \quad \rho = \sqrt{\frac{c-v}{c+v}}\,.$$

   Hence, for an incident wave $\mathbf{E}(\mathbf{x}, t) = F(t_-)\,\mathbf{e}_y$, find the wave that is reflected after it hits a perfectly conducting mirror moving with speed v in the x-direction.

9. (a) A scalar field Φ obeys the wave equation $\partial^\mu\partial_\mu\Phi = 0$. Its *energy-momentum tensor* is $T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi$. Show that $T_{\mu\nu}$ is *conserved*: $\partial_\nu T^{\mu\nu} = 0$.

   (b) The energy-momentum tensor of the Maxwell field is $T_{\mu\nu} = \mu_0^{-1}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$. Explain how T_{00} and T_{0i} are related to the energy density and Poynting vector of the electromagnetic field. Show that Maxwell's equations imply that $\partial_\nu T^{\mu\nu} = -F^\mu{}_\nu j^\nu$ and that the time component of this equation is the energy conservation equation for Maxwell's theory.

10. ($\star$) For a general 4-velocity, written as $U^\mu = \gamma(c, \mathbf{v})$, show that

$$F^{\mu\nu}U_\nu = \gamma\begin{pmatrix} \mathbf{E}\cdot\mathbf{v}/c \\ \mathbf{E} + \mathbf{v}\times\mathbf{B} \end{pmatrix}.$$

   In the rest frame of a conducting medium, Ohm's law states that $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity and $\mathbf{J}$ is the 3-current. Assuming that σ is a Lorentz scalar, show that Ohm's law can be written covariantly as

$$j^\mu + \frac{1}{c^2}(j^\nu U_\nu)U^\mu = \sigma F^{\mu\nu}U_\nu\,,$$

   where j^μ is the charge-current density and U^μ is the (uniform) 4-velocity of the medium. If the medium moves with 3-velocity $\mathbf{v}$ in some inertial frame, show that the current in that frame is

$$\mathbf{J} = \rho\mathbf{v} + \sigma\gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{E})\mathbf{v}\right),$$

   where ρ is the charge density. Simplify this formula, given that the charge density vanishes in the rest frame of the medium.

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