Example Sheet 1

1. A steady two-dimensional flow (pure straining) is given by \( u = \alpha x, \) \( v = -\alpha y \) with \( \alpha \) constant.
   (i) Find the equation for a general streamline of the flow, and sketch some of them.
   (ii) At \( t = 0 \) the fluid on the curve \( x^2 + y^2 = a^2 \) is marked (by an electro-chemical technique). Find the equation for this material fluid curve for \( t > 0 \).
   (iii) Does the area within the curve change in time, and why?

2. Repeat question 1 for the two-dimensional flow (simple shear) given by \( u = \gamma y, \) \( v = 0 \) with \( \gamma \) constant. Which of the two flows stretches the curve faster at long times?

3. A two-dimensional flow is represented by a streamfunction \( \psi(x, y) \) with \( u = \partial \psi/\partial y \) and \( v = -\partial \psi/\partial x \). Show that
   (i) the streamlines are given by \( \psi = \text{const} \),
   (ii) \( |u| = |\nabla \psi| \), so that the flow is faster where the streamlines are closer,
   (iii) the volume flux crossing any curve from \( x_0 \) to \( x_1 \) is given by \( \psi(x_1) - \psi(x_0) \),
   (iv) \( \psi = \text{const} \) on any fixed (i.e. stationary) boundary.
   [Hint for (iii): \( n \cdot ds = (dy, -dx) \).]

4. Verify that the two-dimensional flow given in Cartesian coordinates by
   
   \[
   u = \frac{y - b}{(x - a)^2 + (y - b)^2}, \quad v = \frac{a - x}{(x - a)^2 + (y - b)^2}
   \]

   satisfies \( \nabla \cdot u = 0 \), and then find the streamfunction \( \psi(x, y) \) such that \( u = \partial \psi/\partial y \) and \( v = -\partial \psi/\partial x \). Sketch the streamlines.

5. Verify that the two-dimensional flow given in polar coordinates by
   
   \[
   u_r = U \left( 1 - \frac{a^2}{r^2} \right) \cos \theta, \quad u_\theta = -U \left( 1 + \frac{a^2}{r^2} \right) \sin \theta
   \]

   satisfies \( \nabla \cdot u = 0 \), and find the streamfunction \( \psi(r, \theta) \). Sketch the streamlines.

   \[
   \begin{bmatrix}
   \text{Take:} \quad \nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (u_\theta) \quad \text{and} \quad u_r = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad u_\theta = -\frac{\partial \psi}{\partial \theta}
   \end{bmatrix}
   \]

6. Verify that the axisymmetric flow (uniaxial straining) given in cylindrical polar coordinates by \( u_r = -\frac{1}{2} \alpha r, \) \( u_z = \alpha z \) satisfies \( \nabla \cdot \mathbf{u} = 0 \), and find the Stokes streamfunction \( \Psi(r, z) \). Sketch the streamlines.

   \[
   \begin{bmatrix}
   \text{Take:} \quad \nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z} \quad \text{and} \quad u_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad u_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}
   \end{bmatrix}
   \]
7. Consider the two-dimensional flow \( u = 1/(1 + t), \ v = 1 \) in \( t > -1 \). Find and sketch 
(i) the streamline at \( t = 0 \) which passes through the point \((1, 1)\),
(ii) the path of a fluid particle which is released from \((1, 1)\) at \( t = 0 \).

8. An axisymmetric jet of water of speed \( 1 \) m s\(^{-1}\) and cross-section \( 6 \times 10^{-4} \) m\(^2\) strikes a wall at right angles and spreads out over it. By using the momentum integral equation, calculate the force on the wall due to the jet. [Neglect gravity.]

9. Starting from the Euler momentum equation for a fluid of constant density with a potential force \(-\nabla \chi\), show that for a fixed volume \( V \) enclosed by surface \( A \)

\[
\frac{d}{dt} \int_V \frac{1}{2} \rho u^2 \, dV + \int_A H \cdot \mathbf{n} \, dA = 0
\]

where \( H = \frac{1}{2} \rho u^2 + p + \chi \) is the Bernoulli quantity, so concluding that \( H \) is the transportable energy.

10. How high can water rise up one’s arm hanging in the river from a lazy (1 m s\(^{-1}\)) punt? [Use Bernoulli on surface streamline, where \( p = 1 \) atmosphere.]

11. A rotating circular tank of radius \( a \) is filled with a volume \( V \) of fluid of density \( \rho \). The tank is allowed to rotate for a long time, until the flow is a two-dimensional rigid-body motion with constant angular velocity \( \omega \) around the axis (so that \( u = -\omega y, \ v = \omega x, \ w = 0, \ \rho = \text{const} \)). Derive the equation for the pressure, \( p \), at any point in the rotating fluid. What is the equation for the height \( h(r) \) of the free surface? (Hint: Integrate the Euler equation to find the pressure and determine the constant of integration from the volume conservation).

12. Waste water flows into a tank at \( 10^{-4} \) m\(^3\) s\(^{-1}\) and out of a short exit pipe of cross-section \( 4 \times 10^{-5} \) m\(^2\) into the air. In steady state, estimate how high above the pipe is the water in the tank?

13. A water clock is an axisymmetric vessel with a small exit pipe in the bottom. Find the shape for which the water level falls equal heights in equal intervals of time.