Fluid Dynamics IB
Lent 2017
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Example Sheet 1

Note to students and supervisors: Every answer should include a relevant sketch.

1. By considering the forces acting on a slab of incompressible, viscous fluid undergoing an unsteady parallel shear flow $\mathbf{u} = (u(y,t), 0, 0)$ acted on by a body force (force per unit volume) $\mathbf{f} = (f_x, f_y, 0)$, in Cartesian coordinates (x, y, z), show that

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x,$$
$$0 = -\frac{\partial p}{\partial y} + f_y.$$

2. A film of viscous fluid of uniform thickness h flows steadily under the influence of gravity down a rigid vertical wall. Assume that the surrounding air exerts no stress on the fluid. Calculate the velocity profile and find the volume flux (per unit width) of fluid down the wall.

3. A long, horizontal, two-dimensional container of depth h, filled with viscous fluid, has rigid, stationary bottom and end walls and a rigid top wall that moves with velocity $(U, 0)$ in Cartesian coordinates (x, y), where x is horizontal and U is constant. Assume that the fluid flow far from the end walls is parallel and steady with components $(u(y), 0)$. Determine $u(y)$ and hence determine the tangential force per unit area exerted by the fluid on each of the top and bottom walls.

[*Hint: No penetration through the end walls demands that the volume flux across any vertical cross-section of the flow is zero, which determines the horizontal pressure gradient.*]

4. A two-dimensional, semi-infinite layer of viscous fluid lies above a rigid boundary at $y = 0$ that is oscillating in its own plane with velocity $(U_0 \cos \omega t, 0)$. Assume that there is no pressure gradient and that the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$. By writing $u(y,t) = Re[U_0 f(y) e^{i\omega t}]$ or otherwise, show that

$$f(y) = \exp\left[-(1+i)\sqrt{\frac{\omega}{2\nu}}\, y\right]$$

and hence that the velocity decays within a characteristic distance of $\sqrt{\nu/\omega}$ from the boundary. Calculate the shear stress on the boundary and hence calculate the mean rate of doing work per unit area of the boundary.

5. An infinite horizontal layer of viscous fluid of depth h is initially stationary and has a rigid, stationary upper boundary while its lower, rigid boundary is set into parallel motion with constant speed U at time $t = 0$. Write down the equation, the initial condition and the boundary conditions satisfied by the subsequent flow $(u(y,t), 0)$. What is the steady flow $u_\infty(y)$ that is established after a long time? By writing $u(y,t) = u_\infty(y) - \hat{u}(y,t)$ and using separation of variables, determine a series solution for the transient flow $\hat{u}$. Show that the shear stress exerted by the fluid on the boundary at $y = 0$ is divergent as $t \to 0^+$ but that it is subsequently finite and tends to $-\mu U/h$ as $t \to \infty$.

6. Consider the two-dimensional flow $u = 1/(1+t)$, $v = 1$ in $t > 0$. Find and sketch
(i) the streamline at $t = 0$ which passes through the point $(1, 1)$,
(ii) the path of a fluid particle which is released from $(1, 1)$ at $t = 0$,
(iii) the position at $t = 0$ of a streak of dye released from $(1, 1)$ during the time interval $-1 < t \leq 0$.

7. A two-dimensional flow is represented by a streamfunction $\psi(x, y)$ with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Without invoking the no-slip condition, show that
(i) the streamlines are given by $\psi =$ const,
(ii) $|\mathbf{u}| = |\nabla\psi|$, so that the flow is faster where the streamlines are closer,
(iii) the volume flux crossing any curve from $\mathbf{x}_0$ to $\mathbf{x}_1$ is given by $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$,
(iv) $\psi =$ const on any *fixed* (i.e. stationary) boundary.
[*Hint for (iii):* $\mathbf{n}\, ds = (dy, -dx)$.]

8. Verify that the two-dimensional flow given in Cartesian coordinates by

$$u = \frac{y-b}{(x-a)^2 + (y-b)^2}, \qquad v = \frac{a-x}{(x-a)^2 + (y-b)^2}$$

satisfies $\nabla\cdot\mathbf{u} = 0$, and then find the streamfunction $\psi(x, y)$ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Sketch the streamlines. Calculate (*i*) $\nabla\times\mathbf{u}$ and (*ii*) $\oint_C \mathbf{u}\cdot\mathbf{dl}$ where C is the circle $(x-a)^2 + (y-b)^2 = c^2$. Are these results consistent with Stokes' Theorem?

9. Verify that the two-dimensional flow given in polar coordinates by

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

satisfies $\nabla\cdot\mathbf{u} = 0$, and find the streamfunction $\psi(r, \theta)$. Sketch the streamlines, starting with $\psi = 0$.

$$\left[\text{Note:} \quad \nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta) \quad \text{and take} \quad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r}\right]$$

10. A steady two-dimensional flow (pure straining) is given by $u = \alpha x$, $v = -\alpha y$ with $\alpha > 0$ and constant.
(i) Find the equation for a general streamline of the flow, and sketch some of them.
(ii) At $t = 0$ the fluid on the curve $x^2 + y^2 = a^2$ is marked (by an electro-chemical technique). Find the equation for this material fluid curve for $t > 0$.
(iii) Does the area within the curve change in time, and why?

11. Repeat question 10 for the two-dimensional flow (simple shear) given by $u = \gamma y$, $v = 0$ with $\gamma > 0$ and constant. Which of the two flows stretches the curve faster at long times?

Please email corrections/comments to ngb23@cam.ac.uk