1. A steady two-dimensional flow (pure straining) is given by $u = \alpha x$, $v = -\alpha y$ with $\alpha$ constant.
   (i) Find the equation for a general streamline of the flow, and sketch some of them.
   (ii) At $t = 0$ the fluid on the curve $x^2 + y^2 = a^2$ is marked (by an electro-chemical technique). Find the equation for this material fluid curve for $t > 0$.
   (iii) Does the area within the curve change in time, and why?

2. Repeat question 1 for the two-dimensional flow (simple shear) given by $u = \gamma y$, $v = 0$ with $\gamma$ constant. Which of the two flows stretches the curve faster at long times?

3. A two-dimensional flow is represented by a streamfunction $\psi(x, y)$ with $u = \partial \psi/\partial y$ and $v = -\partial \psi/\partial x$. Show that
   (i) the streamlines are given by $\psi$ = const,
   (ii) $|u| = |\nabla \psi|$, so that the flow is faster where the streamlines are closer,
   (iii) the volume flux crossing any curve from $x_0$ to $x_1$ is given by $\psi(x_1) - \psi(x_0)$,
   (iv) $\psi$ = const on any fixed (i.e. stationary) boundary.
   [Hint for (iii): $n \, ds = (dy, -dx)$]

4. Verify that the two-dimensional flow given in Cartesian coordinates by
   
   $$
   u = \frac{y - b}{(x - a)^2 + (y - b)^2}, \quad v = \frac{a - x}{(x - a)^2 + (y - b)^2}
   $$
   
   satisfies $\nabla \cdot u = 0$, and then find the streamfunction $\psi(x, y)$ such that $u = \partial \psi/\partial y$ and $v = -\partial \psi/\partial x$. Sketch the streamlines.

5. Verify that the two-dimensional flow given in polar coordinates by
   
   $$
   u_r = U \left(1 - \frac{a^2}{r^2}\right) \cos \theta, \quad u_\theta = -U \left(1 + \frac{a^2}{r^2}\right) \sin \theta
   $$
   
   satisfies $\nabla \cdot u = 0$, and find the streamfunction $\psi(r, \theta)$. Sketch the streamlines.

   [Take: $\nabla \cdot u = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (u_\theta)$ and $u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$, $u_\theta = -\frac{\partial \psi}{\partial r}$]

6. Verify that the axisymmetric flow (uniaxial straining) given in cylindrical polar coordinates by $u_r = -\frac{1}{2} \alpha r$, $u_z = \alpha z$ satisfies $\nabla \cdot u = 0$, and find the Stokes streamfunction $\Psi(r, z)$. Sketch the streamlines.

   [Take: $\nabla \cdot u = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z}$ and $u_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}$, $u_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}$]
7. Consider the two-dimensional flow \( u = 1/(1 + t) \), \( v = 1 \) in \( t > -1 \). Find and sketch (i) the streamline at \( t = 0 \) which passes through the point \( (1,1) \), (ii) the path of a fluid particle which is released from \( (1,1) \) at \( t = 0 \).

8. An axisymmetric jet of water of speed \( 1 \text{ m s}^{-1} \) and cross-section \( 6 \times 10^{-4} \text{ m}^2 \) strikes a wall at right angles and spreads out over it. By using the momentum integral equation, calculate the force on the wall due to the jet. [Neglect gravity.]

9. Starting from the Euler momentum equation for a fluid of constant density with a potential force \( -\nabla \chi \), show that for a fixed volume \( V \) enclosed by surface \( A \)

\[
\frac{d}{dt} \int_V \frac{1}{2} \rho u^2 \, dV + \int_A Hu \cdot n \, dA = 0
\]

where \( H = \frac{1}{2} \rho u^2 + p + \chi \) is the Bernoulli quantity, so concluding that \( H \) is the transportable energy.

10. How high can water rise up one's arm hanging in the river from a lazy \( (1 \text{ m s}^{-1}) \) punt? [Use Bernoulli on surface streamline, where \( p = 1 \text{ atmosphere} \).]

11. A rotating circular tank of radius \( a \) is filled with a volume \( V \) of fluid of density \( \rho \). The tank is allowed to rotate for a long time, until the flow is a two-dimensional rigid-body motion with constant angular velocity \( \omega \) around the axis (so that \( u = -\omega y \), \( v = \omega x \), \( w = 0 \), \( \rho = \text{const} \)). Derive the equation for the pressure, \( p \), at any point in the rotating fluid. What is the equation for the height \( h(r) \) of the free surface? (Hint: Integrate the Euler equation to find the pressure and determine the constant of integration from the volume conservation).

12. Waste water flows into a tank at \( 10^{-4} \text{ m}^3 \text{s}^{-1} \) and out of a short exit pipe of cross-section \( 4 \times 10^{-5} \text{ m}^2 \) into the air. In steady state, estimate how high above the pipe is the water in the tank?

13. A water clock is an axisymmetric vessel with a small exit pipe in the bottom. Find the shape for which the water level falls equal heights in equal intervals of time.

Please email corrections/comments to N.G.Berloff@damtp.cam.ac.uk