Fluid Dynamics IB
Lent 2012
Professor M G Worster

Example Sheet 2

Note to students and supervisors: Every answer should include a relevant sketch.

1. An axisymmetric jet of water of speed $1\,\mathrm{m\,s^{-1}}$ and cross-section $6 \times 10^{-4}\,\mathrm{m}^2$ strikes a wall at right angles and spreads out over it. By using the momentum integral equation and neglecting gravity, calculate the force on the wall due to the jet.

2. Starting from the Euler momentum equation for a fluid of constant density with a potential force $-\nabla\chi$, show that for a fixed volume V enclosed by surface A

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2\,dV \;+\; \int_A H\mathbf{u}\cdot\mathbf{n}\,dA \;=\; 0$$

where $H = \frac{1}{2}\rho u^2 + p + \chi$ is the Bernoulli quantity, so concluding that H is the energy transported by the flow.

3. How high can water rise up one's arm hanging in the river from a lazy ($1\,\mathrm{m\,s^{-1}}$) punt? [*Hint: Use Bernoulli on surface streamline.*]

4. Waste water flows into a tank with volume flux Q and out of a short exit pipe of cross-sectional area A into the air. In steady state, how high above the pipe is the water in the tank?

5. A flat-bottomed barge moves very slowly through a closely fitting canal but generates a significant velocity U in the small gap beneath its bottom. Estimate how much lower the barge sits in the water compared to when it is stationary if $U = 5$ m s^{-1}.

6. Water from a large deep reservoir of depth D flows over a broad weir. The water is of depth $d \ll D$ where the free surface has fallen to a level h below that far upstream in the reservoir. Assume that the depth of water varies sufficiently slowly that the velocity is horizontal and uniform in depth. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2gh}$. From the condition that Q does not vary along the flow, and the condition that $h + d$ is a minimum at the crest of the weir [differentiate], show that $h = \frac{1}{2}d$ at the crest. Deduce that $Q^2 = 8gL^3/27$ where L is the minimum value of $h + d$.

7. Using a Taylor expansion, show that, to leading order in small $\delta\mathbf{x}$, $\mathbf{u}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{u}(\mathbf{x})$ can be written in suffix notation as $(E_{ij} + \frac{1}{2}\varepsilon_{jik}\omega_k)\delta x_j$ where $E_{ij} = E_{ji}$ and $\omega_k = (\nabla \times \mathbf{u})_k$. Find E_{ij} and ω_k for the case of linear shear flow $\mathbf{u} = (y, 0, 0)$ and sketch the streamlines of the flows $(\mathbf{u}_1)_i = E_{ij}x_j$ and $(\mathbf{u}_2)_i = \frac{1}{2}\varepsilon_{jik}\omega_k x_j$ for this case.

8. Calculate the vorticity of the velocity field

$$u = -\alpha x - yrf(t), \quad v = -\alpha y + xrf(t), \quad w = 2\alpha z$$

where $r^2 = x^2 + y^2$. Use the vorticity equation to deduce that $f(t) \propto \mathrm{e}^{3\alpha t}$. Explain the nature of this flow and describe the physical principle illustrated by your result.

9. If $\mathbf{u} = \mathbf{\Omega} \times \mathbf{x}$ (uniform rotation with angular velocity $\mathbf{\Omega}$) show that $\boldsymbol{\omega} = 2\mathbf{\Omega}$.

For a two-dimensional flow $\big(u(x,y), v(x,y), 0\big)$ show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, where ψ is the stream function.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes a and b. While $t < 0$ both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0, 0, \Omega)$. What is the vorticity of the flow? Sketch the streamlines noting that they intersect the elliptical boundary of the cylinder. (Why?).

At $t = 0$ the cylinder is suddenly brought to rest. What is the vorticity for $t > 0$? Verify that the flow can be described by

$$\psi = \frac{a^2 b^2 \Omega}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

in suitable coordinates and sketch the streamlines.

10. A sphere of radius a moves with constant velocity U in a fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$? Show that the acceleration of a fluid particle at distance x ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4} - \frac{a^6}{x^7}\right).$$

11. Write down the velocity potential $\phi(x,y)$ for the two-dimensional flow produced by a point source of strength m located at the origin in a uniform stream $(U, 0)$. Show that there is a stagnation point at $(-a, 0)$, where $a = m/2\pi U$. Sketch the streamlines. Show that the streamfunction is given by $\psi = Uy + Ua\theta$, where θ is the polar angle from the positive x axis. From the sketch and the streamfunction show that ϕ represents the flow past a semi-infinite body whose width tends to $2\pi a$ far downstream.

Please email corrections/comments to mgw1@cam.ac.uk