Example Sheet 2

1. By considering the forces acting on a small rectangular slab of incompressible, viscous fluid undergoing an unsteady parallel shear flow $\mathbf{u} = (u(y,t), 0, 0)$, including a body force $\mathbf{F} = (f_x, f_y, 0)$, show that

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x,$$
$$0 = -\frac{\partial p}{\partial y} + f_y.$$

2. A film of viscous fluid of uniform thickness h flows steadily under the influence of gravity down a rigid vertical wall. Assume that the surrounding air exerts only a constant pressure on the fluid. Calculate the velocity profile and find the volume flux (per unit width) of fluid down the wall.

3. A long, horizontal, two-dimensional container of uniform depth h, filled with viscous fluid, has stationary, rigid bottom and end walls, and a rigid top that moves with constant velocity $(U, 0)$ in Cartesian coordinates (x, y). Assume that the flow far from the end walls is parallel and steady with components $(u(y), 0)$. Determine $u(y)$ and hence determine the tangential stress exerted by the fluid on each of the top and bottom boundaries. Describe the overall force balance on a section $x_0 < x < x_1$ of the flow.

[*Hint: No penetration through the end walls requires the volume flux across any vertical cross-section of the flow to be zero, which determines the horizontal pressure gradient.*]

4. A two-dimensional, semi-infinite layer of viscous fluid lies above a rigid boundary at $y = 0$ that oscillates in its own plane with velocity $(U_0 \cos \omega t, 0)$. There is no applied pressure gradient and the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$. By writing $u(y,t) = \mathrm{Re}\,[U_0 f(y) e^{i\omega t}]$,† show that

$$f(y) = \exp\left[-(1+i)\sqrt{\frac{\omega}{2\nu}}\,y\right]$$

and hence that the velocity decays away from the boundary over a characteristic lengthscale $\sqrt{\nu/\omega}$. Sketch the velocity profile at $t = 0$ and $\omega t = \pi/2$.

Calculate the shear stress on the boundary and hence calculate the mean rate of doing work (per unit area) by the boundary.

† *Real part, not Reynolds number!*

5. An infinite layer of viscous fluid of depth h is initially stationary and has a stationary rigid upper boundary, while its rigid lower boundary is set into parallel motion with constant speed U at time $t = 0$. Write down the equation, the initial condition and the boundary conditions satisfied by the subsequent flow $(u(y,t), 0)$. What is the steady flow $u_\infty(y)$ that is established after a long time? By writing $u(y,t) = u_\infty(y) - \hat{u}(y,t)$ and using separation of variables, determine a series solution for the transient flow $\hat{u}$. Show that the shear stress exerted by the fluid on the boundary at $y = 0$ is divergent as $t \to 0^+$ but that it is subsequently finite and tends to $-\mu U/h$ as $t \to \infty$.

6. Water from a large deep reservoir flows steadily over a long straight weir with a wide crest. Over the weir, the water is of depth $d(x)$ and the free surface has fallen to a level $h(x)$ below that far upstream in the reservoir. Assume that the depth of water varies sufficiently slowly that the velocity is nearly horizontal and uniform in depth, and that dh/dx is non-zero at the crest of the weir. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2gh}$. From the condition that Q does not vary along the flow, and the condition that $h + d$ is a minimum at the crest of the weir [differentiate], show that $h = \frac{1}{2}d$ at the crest. Deduce that $Q^2 = 8gL^3/27$ where L is the minimum value of $h + d$.

7. Calculate the vorticity of the velocity field

$$u = -\alpha x - yrf(t), \quad v = -\alpha y + xrf(t), \quad w = 2\alpha z$$

where $r^2 = x^2 + y^2$. Use the (inviscid) vorticity equation to deduce that $f(t) \propto \mathrm{e}^{3\alpha t}$. Explain the nature of this flow and describe the physical principle illustrated by your result. (Why is the growth rate 3α?)

8. If $\mathbf{u} = \boldsymbol{\Omega} \wedge \mathbf{x}$ (uniform rotation with angular velocity $\boldsymbol{\Omega}$) show that $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$.

For a two-dimensional flow $\big(u(x,y), v(x,y), 0\big)$ show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, where ψ is the streamfunction.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes a and b. While $t < 0$ both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0, 0, \Omega)$. What is the vorticity of the flow? Sketch the streamlines noting that they intersect the elliptical boundary of the cylinder. (Why?).

At $t = 0$ the cylinder is suddenly brought to rest. What is the vorticity for $t > 0$? Verify that the flow can be described by

$$\psi = \frac{a^2 b^2 \Omega}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

in suitable coordinates and sketch the streamlines.

9. A sphere of radius a moves with constant velocity U through inviscid fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$? Show that the acceleration of a fluid particle at distance x ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4} - \frac{a^6}{x^7}\right).$$

10. Write down the velocity potential $\phi(x, y)$ for the two-dimensional flow produced by a point source of strength q located at the origin in a uniform stream $(U, 0)$. Show that there is a stagnation point at $(-a, 0)$, where $a = q/2\pi U$. Sketch the streamlines. Show that the streamfunction is given by $\psi = Uy + Ua\theta$, where θ is the polar angle from the positive x axis. From the sketch and the streamfunction show that ϕ represents the flow past a semi-infinite body whose width tends to $2\pi a$ far downstream.

Please email corrections/comments to lister@damtp.cam.ac.uk