Fluid Dynamics IB
Lent 2017
Professor N G Berloff

Example Sheet 2

Note to students and supervisors: Every answer should include a relevant sketch.

1. How high can water rise up one's arm hanging in the river from a lazy ($1\,\mathrm{m\,s^{-1}}$) punt? [*Hint: Use Bernoulli on surface streamline.*]

2. Waste water flows into an open-topped tank with volume flux Q and out of an exit pipe of small cross-sectional area A into the air. In steady state, how high above the pipe is the water in the tank?

3. A flat-bottomed barge moves very slowly through a closely fitting canal but generates a significant velocity U in the small gap beneath its bottom. Estimate how much lower the barge sits in the water compared to when it is stationary if $U = 5$ m s^{-1}.

4. Water from a large deep reservoir of depth D flows over a broad weir. Over the weir, the water is of depth $d(x) \ll D$ where the free surface has fallen to a level $h(x)$ below that far upstream in the reservoir, where x is downstream distance. Assume that the depth of water varies sufficiently slowly that the velocity is horizontal and uniform in depth and that dh/dx is non-zero at the crest of the weir. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2gh}$. From the condition that Q does not vary along the flow, and the condition that $h+d$ is a minimum at the crest of the weir [differentiate], show that $h = \frac{1}{2}d$ at the crest. Deduce that $Q^2 = 8gL^3/27$ where L is the minimum value of $h + d$.

5. An axisymmetric jet of water of speed $1\,\mathrm{m\,s^{-1}}$ and cross-section $6 \times 10^{-4}\,\mathrm{m^2}$ strikes a wall at right angles and spreads out over it. By using the momentum integral equation over a suitable control volume and neglecting gravity, calculate the force on the wall due to the jet.

6. Starting from the Euler momentum equation for a fluid of constant density with a potential force $-\nabla\chi$, show that for a fixed volume V enclosed by surface A

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2\, dV \;+\; \int_A H\mathbf{u}\cdot\mathbf{n}\, dA \;=\; 0$$

where $H = \frac{1}{2}\rho u^2 + p + \chi$ is the Bernoulli quantity, so concluding that H is the energy transported by the flow.

7. Using a Taylor expansion, show that, to leading order in small $\delta\mathbf{x}$, $\mathbf{u}(\mathbf{x}+\delta\mathbf{x}) - \mathbf{u}(\mathbf{x})$ can be written in suffix notation as $(E_{ij} + \frac{1}{2}\varepsilon_{jik}\omega_k)\delta x_j$ where $E_{ij} = E_{ji}$ and $\omega_k = (\nabla\times\mathbf{u})_k$. Find E_{ij} and ω_k for the case of linear shear flow $\mathbf{u} = (y, 0, 0)$ and sketch the streamlines of the flows $(\mathbf{u}_1)_i = E_{ij}x_j$ and $(\mathbf{u}_2)_i = \frac{1}{2}\varepsilon_{jik}\omega_k x_j$ for this case.

8. Calculate the vorticity of the velocity field

$$u = -\alpha x - yrf(t), \quad v = -\alpha y + xrf(t), \quad w = 2\alpha z$$

where $r^2 = x^2 + y^2$. Use the vorticity equation to deduce that $f(t) \propto e^{3\alpha t}$. Explain the nature of this flow and describe the physical principle illustrated by your result.

9. If $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}$ (uniform rotation with angular velocity $\boldsymbol{\Omega}$) show that $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$.

For a two-dimensional flow $\big(u(x,y), v(x,y), 0\big)$ show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, where ψ is the stream function.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes a and b. While $t < 0$ both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0, 0, \Omega)$. What is the vorticity of the flow? Sketch the streamlines noting that they intersect the elliptical boundary of the cylinder. (Why?).

At $t = 0$ the cylinder is suddenly brought to rest. What is the vorticity for $t > 0$? Verify that the flow can be described by

$$\psi = \frac{a^2 b^2 \Omega}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

in suitable coordinates and sketch the streamlines.

10. A sphere of radius a moves with constant velocity U in a fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$? Show that the acceleration of a fluid particle at distance x ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4} - \frac{a^6}{x^7}\right).$$

11. Write down the velocity potential $\phi(x, y)$ for the two-dimensional flow produced by a point source of strength m located at the origin in a uniform stream $(U, 0)$. Show that there is a stagnation point at $(-a, 0)$, where $a = m/2\pi U$. Sketch the streamlines. Show that the streamfunction is given by $\psi = Uy + Ua\theta$, where θ is the polar angle from the positive x axis. From the sketch and the streamfunction show that ϕ represents the flow past a semi-infinite body whose width tends to $2\pi a$ far downstream.

Please email corrections/comments to ngb23@cam.ac.uk