Fluid Dynamics IB
Lent 2018
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Example Sheet 2

**Note to students and supervisors:** Every answer should include a relevant sketch.

1. By considering the forces acting on a slab of incompressible, viscous fluid undergoing an unsteady parallel shear flow $\mathbf{u} = (u(y,t), 0, 0)$ acted on by a body force (force per unit volume) $\mathbf{f} = (f_x, f_y, 0)$, in Cartesian coordinates $(x, y, z)$, show that

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x,$$
$$0 = -\frac{\partial p}{\partial y} + f_y.$$

2. A film of viscous fluid of uniform thickness $h$ flows steadily under the influence of gravity down a rigid vertical wall. Assume that the surrounding air exerts no stress on the fluid. Calculate the velocity profile and find the volume flux (per unit width) of fluid down the wall.

3. A long, horizontal, two-dimensional container of depth $h$, filled with viscous fluid, has rigid, stationary bottom and end walls and a rigid top wall that moves with velocity $(U, 0)$ in Cartesian coordinates $(x, y)$, where $x$ is horizontal and $U$ is constant. Assume that the fluid flow far from the end walls is parallel and steady with components $(u(y), 0)$. Determine $u(y)$ and hence determine the tangential force per unit area exerted by the fluid on each of the top and bottom walls.

[*Hint: No penetration through the end walls demands that the volume flux across any vertical cross-section of the flow is zero, which determines the horizontal pressure gradient.*]

4. A two-dimensional, semi-infinite layer of viscous fluid lies above a rigid boundary at $y = 0$ that is oscillating in its own plane with velocity $(U_0 \cos\omega t, 0)$. Assume that there is no pressure gradient and that the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$. By writing $u(y,t) = Re[U_0 f(y) e^{i\omega t}]$ or otherwise, show that

$$f(y) = \exp\left[-(1+i)\sqrt{\frac{\omega}{2\nu}}y\right]$$

and hence that the velocity decays within a characteristic distance of $\sqrt{\nu/\omega}$ from the boundary. Calculate the shear stress on the boundary and hence calculate the mean rate of doing work per unit area of the boundary.

5. An infinite horizontal layer of viscous fluid of depth $h$ is initially stationary and has a rigid, stationary upper boundary while its lower, rigid boundary is set into parallel motion with constant speed $U$ at time $t = 0$. Write down the equation, the initial condition and the boundary conditions satisfied by the subsequent flow $(u(y,t), 0)$. What is the steady flow $u_\infty(y)$ that is established after a long time? By writing $u(y,t) = u_\infty(y) - \hat{u}(y,t)$ and using separation of variables, determine a series solution for the transient flow $\hat{u}$. Show that the shear stress exerted by the fluid on the boundary at $y = 0$ is divergent as $t \to 0^+$ but that it is subsequently finite and tends to $-\mu U/h$ as $t \to \infty$.

6. Water from a large deep reservoir of depth $D$ flows over a broad weir. Over the weir, the water is of depth $d(x) \ll D$ where the free surface has fallen to a level $h(x)$ below that far upstream in the reservoir, where $x$ is downstream distance. Assume that the depth of water varies sufficiently slowly that the velocity is horizontal and uniform in depth and that $dh/dx$ is non-zero at the crest of the weir. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2gh}$. From the condition that $Q$ does not vary along the flow, and the condition that $h + d$ is a minimum at the crest of the weir [differentiate], show that $h = \frac{1}{2}d$ at the crest. Deduce that $Q^2 = 8gL^3/27$ where $L$ is the minimum value of $h + d$.

7. Calculate the vorticity of the velocity field

$$u = -\alpha x - yrf(t), \quad v = -\alpha y + xrf(t), \quad w = 2\alpha z$$

where $r^2 = x^2 + y^2$. Use the vorticity equation to deduce that $f(t) \propto \mathrm{e}^{3\alpha t}$. Explain the nature of this flow and describe the physical principle illustrated by your result.

8. If $\mathbf{u} = \mathbf{\Omega} \times \mathbf{x}$ (uniform rotation with angular velocity $\mathbf{\Omega}$) show that $\boldsymbol{\omega} = 2\mathbf{\Omega}$.

For a two-dimensional flow $\big(u(x,y), v(x,y), 0\big)$ show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, where $\psi$ is the stream function.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes $a$ and $b$. While $t < 0$ both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0, 0, \Omega)$. What is the vorticity of the flow? Sketch the streamlines noting that they intersect the elliptical boundary of the cylinder. (Why?).

At $t = 0$ the cylinder is suddenly brought to rest. What is the vorticity for $t > 0$? Verify that the flow can be described by

$$\psi = \frac{a^2 b^2 \Omega}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

in suitable coordinates and sketch the streamlines.

9. A sphere of radius $a$ moves with constant velocity $U$ in a fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$? Show that the acceleration of a fluid particle at distance $x$ ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4} - \frac{a^6}{x^7}\right).$$

10. Write down the velocity potential $\phi(x, y)$ for the two-dimensional flow produced by a point source of strength $m$ located at the origin in a uniform stream $(U, 0)$. Show that there is a stagnation point at $(-a, 0)$, where $a = m/2\pi U$. Sketch the streamlines. Show that the streamfunction is given by $\psi = Uy + Ua\theta$, where $\theta$ is the polar angle from the positive $x$ axis. From the sketch and the streamfunction show that $\phi$ represents the flow past a semi-infinite body whose width tends to $2\pi a$ far downstream.

Please email corrections/comments to ngb23@cam.ac.uk