Fluid Mechanics IB
Lent 2011
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Example Sheet 3

1. Find the frequencies of small amplitude oscillations for the water surface in a vertical cylinder of radius a and of large depth. [You may assume that the velocity potential is given by separable solutions to Laplace's equation of the form $\phi(r, z, t) = J_0(kr)e^{kz-i\omega t}$ for axisymmetric oscillations and $\phi(r, \theta, z, t) = J_n(kr)\cos(n\theta)e^{kz-i\omega t}$ for non-axisymmetric oscillations, where the functions J_0, J_1 etc. are called Bessel functions.] What restriction does the kinematic boundary condition at $r = a$ place on the value of k?

What is the lowest frequency for a cup of tea? Show the sign of the surface displacement in plan view for the three lowest frequency modes. [The first root of $J_0'(z) = 0$ is $z = 3.83$ while the first root of $J_1'(z) = 0$ is $z = 1.84$. See over for a graph of J_0, J_1 and J_2.]

2. A sphere is immersed in an infinite ocean of incompressible fluid of density ρ. Its radius is given by $R(t) = a + b\sin nt$, where a, b and n are constants with $b < a$. The fluid moves radially under no external forces and the constant pressure at infinity is P. If $a \geq 5b$, show that the maximum pressure attained on the surface of the sphere is $P + \rho n^2 b(a - b)$. What is the corresponding formula if $b < a \leq 5b$?

3. A rigid disk of radius R is at a height $h(t)$ above a fixed plane $z = 0$, with fluid filling the gap between them, and $h \ll R$. Neglecting end effects from near the edge of the disk, show that the flow in the gap is described by

$$\phi = \frac{\dot{h}}{2h}\left(z^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2\right)$$

Assuming that the pressure at the edge of the disk is approximately constant, find the pressure distribution under the disk and hence the force on the fixed plane. Explain how the pressure distribution accelerates the radial flow in the cases $\dot{h} > 0$ and $\dot{h} < 0$.

4. A rigid sphere of radius a executes small amplitude oscillations (with a velocity $\mathbf{U}(t)$) while immersed in a fluid contained within a larger concentric fixed sphere of radius b. Find the velocity potential for the induced fluid motion. Neglecting terms quadratic in the amplitude, show that the pressure distribution over the surface of the moving inner sphere is

$$\rho\dot{U}a\cos\theta(a^3 + \tfrac{1}{2}b^3)/(b^3 - a^3)$$

(where θ is the angle with $\mathbf{U}$) and hence find the force exerted by the fluid on it? Why is the force on the outer fixed sphere different? Comment on the case of a tight fit.

5. Water from a large deep reservoir flows over a weir. The water is of depth d where the free surface has fallen to a level h below that far upstream in the reservoir. Assume that the depth of water varies sufficiently slowly that the velocity is horizontal and uniform in depth. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2gh}$. From the condition that Q does not vary along the flow, and the condition that $h + d$ is a minimum at the crest of the weir [differentiate], show that $h = \frac{1}{2}d$ at the crest. Deduce that $Q^2 = 8gL^3/27$ where L is the minimum value of $h + d$.

6. A river flows in a channel of rectangular cross-section with a flat horizontal bottom and a width $w(x)$ which varies slowly along the channel. Far upstream the fluid velocity u takes the value V, the depth of the water is H and the width of the channel W. Taking $u(x)$ to be constant across the channel, show that [mass, Bernoulli]

$$\frac{W}{w} = \frac{u}{V}\left(1 + \tfrac{1}{2}F - \tfrac{1}{2}F\frac{u^2}{V^2}\right) \quad \text{where} \quad F = \frac{V^2}{gH}$$

Sketch this relationship. Observation of the river shows that $u(x)$ is steady and slowly varying and that far downstream $w \to W$ but $u \to U \neq V$. What can be deduced about W/w in the region of varying width? Find U and the downstream depth.

7. Starting from the analysis of a hydraulic jump in your lecture notes, write down the flux of energy (kinetic plus potential) and the rate of working of pressure forces at control surfaces on either side on the jump. The difference between the two sides is the rate of energy dissipation D due to friction (viscosity) in the jump. By eliminating the speed of the jump relative to the fluid behind it $(V - U_2)$, show that

$$D = \frac{\rho g}{4}(V + U_1)\frac{(h_2 - h_1)^3}{h_2}$$

where h_1 and h_2 are, respectively, the fluid depths ahead of and behind the jump and $V + U_1$ is the speed of the jump relative to the fluid ahead of it.

8. Two line vortices are initially at $(0, d)$ and $(0, -d)$ in Cartesian coordinates. Describe the motion of the vortices if their strengths are: (a) $+\kappa$ and $-\kappa$; (b) $+\kappa$ and $+\kappa$; and (c) $+2\kappa$ and $-\kappa$.

Two-dimensional fluid flow contains a line source of strength m fixed at the origin and two freely moving line vortices of strength $\pm\kappa$. Each vortex moves due to the velocity field of the source and the other vortex. Suppose that the vortices are symmetrically placed at $(r(t), \pm\theta(t))$ in polar coordinates. Find $\dot{r}$ and $\dot{\theta}$ and hence show that

$$r \sin\theta = A\mathrm{e}^{-2m\theta/\kappa}$$

where A is a constant. Sketch the vortex paths.