

## Example Sheet 3

**Note to students and supervisors:** Every answer should include a relevant sketch.

1. An orifice in the side of an open vessel containing water leads smoothly into a horizontal tube of uniform cross-section and length  $L$ . The diameter of the tube is small compared with  $L$ , with the horizontal dimensions of the free surface, and with the depth  $h$  of the orifice below the free surface. A plug at the end of the tube is suddenly removed and the water begins to flow. Show, using the expression for the pressure in unsteady irrotational flow, that the outflow velocity at subsequent times  $t$  is approximately

$$\sqrt{2gh} \tanh\left(\frac{t\sqrt{2gh}}{2L}\right).$$

Estimate the time scale for the flow in a garden hose to accelerate to its maximum velocity (Assume that tap pressure is equivalent to  $\rho gh$  with  $h = 5$  m.)

2. A rigid circular disc of mass  $M$ , radius  $R$  and negligible thickness moves vertically along its axis through an incompressible inviscid fluid above a fixed horizontal plane on  $z = 0$ , the disc occupying  $r < R$ ,  $z = h(t)$  in cylindrical polar coordinates  $(r, \theta, z)$ . Assuming that (for  $h \ll R$ ) the flow in the gap  $r < R$ ,  $0 < z < h(t)$  between disc and plane is axisymmetric with horizontal component independent of  $z$ , i.e. the fluid velocity there is  $\mathbf{u} = u_r(r, t) \hat{\mathbf{r}} + u_z(r, z, t) \hat{\mathbf{z}}$ , show that incompressibility and the no-penetration conditions imply

$$\mathbf{u} = \nabla\phi \quad \text{with} \quad \phi = \frac{\dot{h}}{4h} (2z^2 - r^2) .$$

Find the pressure distribution under the disc and, on the assumption that the pressure can be taken to be uniform (in space) over the top of the disc, determine the force on the disc.

3. A rigid sphere of radius  $a$  executes small-amplitude oscillations with velocity  $U(t)$  in the  $z$ -direction about the centre (taken as origin) of a larger fixed sphere of radius  $b$ . By approximating the boundary condition on the smaller sphere, neglecting terms quadratic in the amplitude, find the velocity potential for the induced motion (assumed irrotational) of the fluid that fills the gap between the two spheres and show that the (dynamic) pressure on the surface of the inner sphere is

$$p_0(t) + \frac{a^3 + \frac{1}{2}b^3}{b^3 - a^3} \rho \dot{U} a \cos \theta$$

where  $\theta$  is the angle with the positive  $z$ -direction. Hence find the (dynamic) force exerted by the fluid on the inner sphere. Why is the force on the outer fixed sphere different? Comment on the case of a tight fit.

4. A U-tube consists of two long uniform vertical tubes of different cross-sectional areas  $A_1$ ,  $A_2$  connected at the base by a short tube of large cross-section, and contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height  $h$  above the base. Derive the equation governing the nonlinear oscillations of the displacement  $\zeta(t)$  of the surface in the tube of cross-section  $A_2$

$$(h + r\zeta) \frac{d^2\zeta}{dt^2} + \frac{r}{2} \left( \frac{d\zeta}{dt} \right)^2 + g\zeta = 0 \quad \text{where} \quad r = 1 - A_2/A_1.$$

5. Water fills a square container with sidewalls at  $x = 0, a$ ,  $y = 0, a$  and base at  $z = -h$ . If the ‘free’ surface, which at rest is at  $z = 0$ , displaces to  $z = \zeta(x, y, t)$  and the motion is irrotational, write down the equation and (exact) boundary conditions satisfied by the velocity potential  $\phi(x, y, z, t)$ . For small-amplitude motion [ $|\zeta| \ll h$ ,  $|\zeta_x| \ll 1$ ,  $|\zeta_y| \ll 1$ ] linearise the free-surface conditions, and deduce that there is then a doubly infinite family of separable solutions of the form

$$\phi = f(x)g(y)h(z)e^{-i\omega t}, \quad \zeta = f(x)g(y)e^{-i\omega t},$$

i.e. a ‘normal mode’ oscillating at a single [angular] frequency  $\omega$ . Show the sign of the surface displacement in plan view for each of five lowest frequency modes.

6. Fluid of density  $\rho_1$  occupies the region  $z > 0$  and overlies another fluid of density  $\rho_2$  (with  $\rho_2 > \rho_1$ ), which occupies the region  $z < 0$ . Show that small-amplitude oscillations with the interface displacing to

$$z = \text{Re} \left( \zeta_0 e^{i(kx - \omega t)} \right),$$

where  $k > 0$  and  $k|\zeta_0| \ll 1$ , satisfy the dispersion relation

$$\omega^2 = gk \left( \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right)$$

[Hint: You will need different potentials  $\phi_1$  and  $\phi_2$  for the two regions and should apply the kinematic boundary condition to the flow in each region.]

7. A rotating circular tank with sides at  $r = a$  and base at  $z = 0$  in cylindrical polars  $(r, \theta, z)$  is filled with a volume  $V$  of fluid of density  $\rho$ . The tank is allowed to rotate for a long time, until (through the action of viscosity) the flow is a steady rigid-body motion with constant angular velocity  $\Omega$  around the  $z$ -axis (so that  $u = -\Omega y, v = \Omega x, w = 0$  in cartesian or equivalently  $u_r = 0, u_\theta = \Omega r, u_z = 0$  in cylindrical polars) and the fluid has a ‘free’ surface at  $z = h(r)$  on which the (absolute) pressure takes the constant (atmospheric) value  $p_0$ . Use the Navier-Stokes equations to determine the (absolute) pressure in the fluid, and hence  $h(r)$ , up to a constant and then use volume conservation to determine  $h(r)$  completely. What is the largest value of  $\Omega$  for which the fluid covers the whole base?

8. Wind blows steadily with uniform speed  $U$  from west to east over the United Kingdom. What is the magnitude and direction of the horizontal pressure gradient? Estimate the pressure difference between London and Edinburgh when  $U = 10 \text{ m s}^{-1}$ .

[You may need to look up values for the physical parameters.]

9. Derive the linearized, rotating, shallow-water equations governing the horizontal flow  $(u, v)$  in a layer of inviscid fluid of depth  $h_0 + \eta(x, y, t)$ , where  $\eta \ll h_0$ ,  $f$  is the Coriolis parameter and  $g$  is the acceleration due to gravity,

$$\begin{aligned}\frac{\partial u}{\partial t} - fv &= -g \frac{\partial \eta}{\partial x}, \\ \frac{\partial v}{\partial t} + fu &= -g \frac{\partial \eta}{\partial y}, \\ \frac{\partial \eta}{\partial t} + h_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0.\end{aligned}$$

Consider the flow parallel to a coastline  $x = 0$  with  $u \equiv 0$  and show that

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0 \frac{\partial^2 \eta}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \eta}{\partial t \partial x} + f \frac{\partial \eta}{\partial y} = 0.$$

Find wavelike solutions of the form  $\eta = A(x)B(y - ct)$  for arbitrary functions  $B$ , and determine the constant wave speed  $c$ . Given that  $\eta \rightarrow 0$  as  $x \rightarrow \infty$ , determine the  $x$ -structure function  $A$  and explain why waves can only travel in the negative  $y$  direction.

[*These solutions represent coastally trapped Kelvin waves. They travel southwards along the east coast of England and northwards along the west coast of the Netherlands. One large such wave was responsible for the devastating floods in East Anglia and the Netherlands in 1953.*]