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Example Sheet 3

1. The velocity in the far-field of steady uniform flow past a stationary two-dimensional aerofoil with circulation κ takes the form

$$\mathbf{u} = (U, 0) + \frac{\kappa}{2\pi r}(-\sin\theta, \cos\theta) + O(1/r^2)\,,$$

where the $O(1/r^2)$ dipole term depends on the detailed shape of the object. Determine the pressure $p(r,\theta)$ in the far-field to the same level of approximation. Use the momentum integral equation to show that the aerofoil experiences a force $(0, -\rho U\kappa)$.

2. An orifice in the side of an open vessel containing water leads smoothly into a horizontal tube of uniform cross-section and length L. The diameter of the tube is small compared with L, with the horizontal dimensions of the free surface, and with the depth h of the orifice below the free surface. A plug at the end of the tube is suddenly removed and the water begins to flow. Show, using the expression for the pressure in unsteady irrotational flow, that the outflow velocity at subsequent times t is approximately

$$\sqrt{2gh}\tanh\left(\frac{t\sqrt{2gh}}{2L}\right).$$

Estimate the time scale for the flow in a garden hose to accelerate to its maximum velocity (Assume that tap pressure is equivalent to $\rho g h$ with $h = 5\,\mathrm{m}$.)

3. A rigid circular disc of radius R is at a height $h(t)$ above a fixed horizontal plane $z = 0$, and inviscid incompressible fluid fills the gap $0 < z < h(t)$, $r < R$ between them. Assume that $h \ll R$ and that the axisymmetric flow in the thin gap has radial component $u_r(r,t)$ independent of z. Use conservation of mass and the boundary conditions to deduce that the velocity in the gap is given by

$$\mathbf{u} = \boldsymbol{\nabla}\phi \quad \text{with} \quad \phi = \frac{\dot{h}}{4h}\left(2z^2 - r^2\right).$$

Assuming that the pressure at the edge of the disc is a constant p_0 (as velocities and pressures are much larger in the thin gap than elsewhere), find the pressure distribution in the gap and hence determine the force on the plane due to the motion.

4. A rigid sphere of radius a executes small-amplitude oscillations with velocity $U(t)\mathbf{e}_z$ about the centre $r = 0$ of a larger fixed sphere of radius b. By linearising the boundary condition on the smaller sphere onto $r = a$, find the velocity potential for the induced irrotational motion of fluid that fills the gap between the two spheres and, again neglecting terms quadratic in the amplitude, show that the (dynamic) pressure on the surface of the inner sphere is

$$\frac{a^3 + \frac{1}{2}b^3}{b^3 - a^3}\,\rho\dot{U}a\cos\theta\,,$$

where θ is the angle from $\mathbf{e}_z$. Hence find the force exerted by the fluid on the inner sphere. Why is the force on the outer fixed sphere different? Comment on the case of a tight fit.

5. A U-tube consists of two long uniform vertical tubes of different cross-sectional areas A_1, A_2 connected at the base by a short tube of large cross-section, and contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation governing the nonlinear oscillations of the displacement $\zeta(t)$ of the surface in the tube of cross-section A_2

$$(h + r\zeta)\frac{d^2\zeta}{dt^2} + \frac{r}{2}\left(\frac{d\zeta}{dt}\right)^2 + g\zeta = 0 \quad \text{where} \quad r = 1 - A_2/A_1.$$

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6. Water fills a square container $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$ to an equilibrium depth h. Write down the equation and (exact) boundary conditions for the velocity potential and the motion of the free surface when it is disturbed from equilibrium. Explain how to linearise the free-surface conditions for small-amplitude disturbances. Seek separable solutions proportional to $\exp(-i\omega t)$ to the linearised equations, and thence obtain the frequencies of the 'normal modes'. Show the sign of the surface displacement in plan view for the five lowest frequency modes.

7. Fluid of density ρ_1 occupies the region $z > 0$ and overlies another fluid of density ρ_2 (with $\rho_2 > \rho_1$), which occupies the region $z < 0$. Show that small-amplitude oscillations with interfacial displacement $\zeta(x,t) \propto \exp[i(kx - \omega t)]$, $k > 0$, satisfy the dispersion relation

$$\omega^2 = gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right).$$

[*Hint: You will need different potentials ϕ_1 and ϕ_2 for the two regions and should apply the kinematic boundary condition to the flow in each region.*]

8. Wind blows steadily with uniform speed U from west to east over the United Kingdom. What is the magnitude and direction of the horizontal pressure gradient? Estimate (to 1 s.f.) the pressure difference between London and Edinburgh when $U = 10$ m s^{-1}.

[*You may need to look up values for the physical parameters.*]

9. Derive the linearized, rotating, shallow-water equations governing the horizontal flow (u, v) in a layer of inviscid fluid of depth $h_0 + \eta(x, y, t)$, where $\eta \ll h_0$, f is the Coriolis parameter and g is the acceleration due to gravity,

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x},$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y},$$
$$\frac{\partial \eta}{\partial t} + h_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

Consider flow parallel to a coastline $x = 0$ with $u \equiv 0$ and show that

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0\frac{\partial^2 \eta}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 \eta}{\partial t \partial x} + f\frac{\partial \eta}{\partial y} = 0.$$

Find wave-like solutions of the form $\eta = A(x)B(y - ct)$ for arbitrary functions B, and determine the constant wave speed c. Given $f > 0$ and $\eta \to 0$ as $x \to \infty$, determine the x-structure function A and explain why waves can only travel in the negative y direction.

[*These solutions represent coastally trapped Kelvin waves. They travel southwards along the east coast of England and northwards along the west coast of the Netherlands. One large such wave was responsible for the devastating floods in East Anglia and the Netherlands in 1953.*]

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