Example Sheet 2

1. Show that $\mathbf{u}(\mathbf{x}+\delta\mathbf{x})-\mathbf{u}(\mathbf{x})$ can be written as $(e_{ij}+\frac{1}{2}\varepsilon_{jik}\omega_k)\delta x_j+O(|\delta\mathbf{x}|^2)$, where $e_{ij}=e_{ji}$ and $\omega_k=(\nabla\times\mathbf{u})_k$. Find e_{ij} and ω_k for the case of linear shear flow $\mathbf{u}=(y,0,0)$, and for this case sketch the streamlines of the two flows whose velocity fields are $e_{ij}x_j$ and $\frac{1}{2}\varepsilon_{jik}\omega_k x_j$.

2. Calculate the vorticity $\boldsymbol{\omega}$ of the velocity field

$$u=-\alpha x-yrf(t),\quad v=-\alpha y+xrf(t),\quad w=2\alpha z$$

where $r^2=x^2+y^2$. Show that $\nabla.\mathbf{u}=0$ for any function $f(t)$, and that the vorticity equation $\partial\boldsymbol{\omega}/\partial t+(\mathbf{u}.\nabla)\boldsymbol{\omega}=(\boldsymbol{\omega}.\nabla)\mathbf{u}$ is satisfied if and only if $f(t)\propto \mathrm{e}^{3\alpha t}$.

Calculate the velocity components u_r and u_θ in cylindrical polar coordinates. Consider a material curve that starts at $t=0$ as $z=0$, $r=a_0$. Show that this becomes $z=0$, $r=a_0e^{-\alpha t}$. Verify the circulation theorem for this material curve.

3. If $\mathbf{u}=\boldsymbol{\Omega}\times\mathbf{x}$ (uniform rotation with angular velocity $\boldsymbol{\Omega}$) show that $\boldsymbol{\omega}=2\boldsymbol{\Omega}$, where $\boldsymbol{\omega}$ is the vorticity. For a two-dimensional flow $\big(u(x,y),v(x,y),0\big)$, show that $\boldsymbol{\omega}=(0,0,-\nabla^2\psi)$, where ψ is the streamfunction.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes a and b. While $t<0$, both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0,0,\Omega)$. Sketch the streamlines, noting that they intersect the elliptical boundary of the cylinder. (Why?)

At $t=0$, the cylinder is suddenly brought to rest. Assuming that the flow remains two-dimensional, what does the vorticity equation say about $\nabla^2\psi$ for $t>0$? Verify that the flow for $t>0$ can be described by

$$\psi=\frac{a^2b^2\Omega}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

in suitable coordinates, and sketch the streamlines.

4. Show that velocity potential ϕ cannot have an extreme value at an interior point of the fluid. Deduce that the speed cannot have a maximum value at an interior point. (Hint: ϕ and its partial derivatives satisfy Laplace's equation.)

5. (1) Is the motion incompressible for the flows given by the following velocity potentials: (a) $\phi=C(x^2+y^2)$ (b) $\phi=C(x^2-y^2)$? If so, determine the corresponding stream functions. (2) Is the motion irrotational for the flows given by the following stream functions: (a) $\psi=C(x^2+y^2)$ (b) $\psi=C(x^2-y^2)$? If so, determine the corresponding velocity potentials. Sketch the streamlines for all cases ((1) and (2)) and the lines of constant ϕ where possible.

6. A sphere of radius a moves with constant velocity U in a fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$ relative to the far-field? Show that the acceleration of a fluid particle at distance x ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4}-\frac{a^6}{x^7}\right)\ .$$

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7. In spherical polar coordinates (r, θ, ξ) the velocity potential

$$\phi = -kr^2 P_2(\cos\theta)$$

represents the axisymmetric flow produced by the confluence of two equal and opposite streams. Sketch the stream line pattern in any plane through the axis of symmetry. If now a solid sphere with surface $r = a$ is placed in this flow, how is the velocity potential modified? Hence, determine the distribution of velocity and pressure over the surface of the sphere.

8. Write down the velocity potential $\phi(r, \theta, z)$ for the axisymmetric flow produced by a point source of strength m located at the origin, in the presence of a uniform stream $(0, 0, U)$. Show that there is a stagnation point at $(0, 0, -\frac{1}{2}a)$, where $a = (m/\pi U)^{1/2}$. Sketch the streamlines. Show that the Stokes streamfunction is given by $\Psi = \frac{1}{2}Ur^2 - mz/4\pi(z^2 + r^2)^{1/2}$. From the sketch and the streamfunction show that ϕ represents the flow around a semi-infinite body whose radius tends to a far downstream.

Sketch the flow if the source is re-located at $(0, 0, -l)$ and a sink of equal strength introduced at $(0, 0, l)$.

9. An orifice in the side of a large open vessel full of water leads smoothly into a horizontal tube of uniform cross-section and length L. The diameter of the tube is small in comparison with L and with the size of the vessel and the depth h of the orifice below the free surface. A plug at the end of the tube is suddenly removed and the water begins to flow. Neglecting small changes in h, show from the relevant form of Bernoulli's theorem, or otherwise, that the outflow velocity at subsequent times t is approximately

$$\sqrt{2gh}\tanh\left(\frac{t\sqrt{2gh}}{2L}\right) .$$

Deduce that the time it takes for the flow to accelerate to a fraction $(e^2 - 1)/(e^2 + 1) = 0.7616$ of its limiting value is $2L/\sqrt{(2gh)}$, and verify that this time is about 3 seconds for a garden hose $L = 9$ m long supplied by a rainwater tank with $h = 1.8$ m.

10. A U-tube consists of two long uniform vertical tubes of different cross-sectional areas A_1, A_2 connected at the base by a short tube of large cross-section, and contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation governing the nonlinear oscillations of the displacement ζ of the surface in the tube of cross-section A_2

$$(h + r\zeta)\frac{d^2\zeta}{dt^2} + \frac{r}{2}\left(\frac{d\zeta}{dt}\right)^2 + g\zeta = 0 \quad \text{where} \quad r = 1 - A_2/A_1 .$$

[Hint: Take $\phi = 0$ at the bottom of the U-tube, and remember that the irrotational form of Bernoulli's theorem involves $\partial\phi/\partial t$, implying time differentiation at a fixed location.]

11. A sphere is immersed in an infinite calm ocean of incompressible fluid of density ρ. Its radius is given by $R(t) = a + b\sin nt$, where a, b and n are constants with $b < a$. The fluid moves radially under no external forces and the constant pressure at infinity is P. If $a \geq 5b$, show that the maximum pressure attained on the surface of the sphere is $P + \rho n^2 b(a - b)$. What is the corresponding formula if $b < a \leq 5b$?

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