

## Numerical Analysis – Lecture 8

### 6.3 The error of polynomial interpolation

Let  $[a, b]$  be a closed interval of  $\mathbb{R}$ . We denote by  $C[a, b]$  the space of all continuous functions from  $[a, b]$  to  $\mathbb{R}$  and let  $C^s[a, b]$ , where  $s$  is a positive integer, stand for the linear space of all functions in  $C[a, b]$  that possess  $s$  continuous derivatives.

**Theorem** Given  $f \in C^{n+1}[a, b]$ , let  $p \in \mathbb{P}_n[x]$  interpolate the values  $f(x_i)$ ,  $i = 0, 1, \dots, n$ , where  $x_0, \dots, x_n \in [a, b]$  are pairwise distinct. Then for every  $x \in [a, b]$  there exists  $\xi \in [a, b]$  such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^n (x - x_i). \quad (6.1)$$

**Proof.** The formula (6.1) is true when  $x = x_j$  for  $j \in \{0, 1, \dots, n\}$ , since both sides of the equation vanish. Let  $x \in [a, b]$  be any other point and define

$$\phi(t) := [f(t) - p(t)] \prod_{i=0}^n (x - x_i) - [f(x) - p(x)] \prod_{i=0}^n (t - x_i), \quad t \in [a, b].$$

[Note: The variable in  $\phi$  is  $t$ , whereas  $x$  is a fixed parameter.] Note that  $\phi(x_j) = 0$ ,  $j = 0, 1, \dots, n$ , and  $\phi(x) = 0$ . Hence,  $\phi$  has at least  $n + 2$  distinct zeros in  $[a, b]$ . Moreover,  $\phi \in C^{n+1}[a, b]$ .

We now apply the *Rolle theorem*: if the function  $g \in C^1[a, b]$  vanishes at two distinct points in  $[a, b]$  then its derivative vanishes at an intermediate point. We deduce that  $\phi'$  vanishes at (at least)  $n + 1$  distinct points in  $[a, b]$ . Next, applying Rolle to  $\phi'$ , we conclude that  $\phi''$  vanishes at  $n$  points in  $[a, b]$ . In general, we prove by induction that  $\phi^{(s)}$  vanishes at  $n + 2 - s$  distinct points of  $[a, b]$  for  $s = 0, 1, \dots, n + 1$ . Letting  $s = n + 1$ , we have  $\phi^{(n+1)}(\xi) = 0$  for some  $\xi \in [a, b]$ . But  $p^{(n+1)} \equiv 0$ ,  $d^{n+1} \prod_{i=0}^n (t - x_i) / dt^{n+1} \equiv (n + 1)!$ , and we obtain (6.1).  $\square$

**Runge's example** We interpolate  $f(x) = 1/(1 + x^2)$ ,  $x \in [-5, 5]$ , at the equally-spaced points  $x_j = -5 + 10\frac{j}{n}$ ,  $j = 0, 1, \dots, n$ . Some of the errors for  $n = 20$  are

$x$	$f(x) - p(x)$	$\prod_{i=0}^n (x - x_i)$
0.75	$3.2 \times 10^{-3}$	$-2.5 \times 10^6$
1.75	$7.7 \times 10^{-3}$	$-6.6 \times 10^6$
2.75	$3.6 \times 10^{-2}$	$-4.1 \times 10^7$
3.75	$5.1 \times 10^{-1}$	$-7.6 \times 10^8$
4.75	$4.0 \times 10^{+2}$	$-7.3 \times 10^{10}$

The growth in the error is explained by the product term in (6.1) (the rightmost column of the table). Adding more interpolation points makes the largest error even worse. A remedy to this state of affairs is to cluster points toward the end of the range.

A considerably smaller error is attained for  $x_j = 5 \cos \frac{(n-j)\pi}{n}$ ,  $j = 0, 1, \dots, n$  (so-called *Chebyshev points*). It is possible to prove that this choice of points minimizes the magnitude of  $\max_{x \in [-5, 5]} |\prod_{i=0}^n (x - x_i)|$ .

### 6.4 Divided differences: a definition

Given pairwise-distinct points  $x_0, x_1, \dots, x_n \in [a, b]$ , we let  $p \in \mathbb{P}_n[x]$  interpolate  $f \in C[a, b]$  there. The coefficient of  $x^n$  in  $p$  is called the *divided difference* and denoted by  $f[x_0, x_1, \dots, x_n]$ . We say

that this divided difference is of *degree*  $n$ .

We can derive  $f[x_0, \dots, x_n]$  from the Lagrange formula,

$$f[x_0, x_1, \dots, x_n] = \sum_{k=0}^n f(x_k) \prod_{\substack{\ell=0 \\ \ell \neq k}}^n \frac{1}{x_k - x_\ell}. \quad (6.2)$$

**Theorem** Let  $[\bar{a}, \bar{b}]$  be the shortest interval that contains  $x_0, x_1, \dots, x_n$  and let  $f \in C^n[\bar{a}, \bar{b}]$ . Then there exists  $\xi \in [\bar{a}, \bar{b}]$  such that

$$f[x_0, x_1, \dots, x_n] = \frac{1}{n!} f^{(n)}(\xi). \quad (6.3)$$

**Proof.** Let  $p$  be the interpolating polynomial. The error function  $f - p$  has at least  $n + 1$  zeros in  $[\bar{a}, \bar{b}]$  and, applying Rolle's theorem  $n$  times, it follows that  $f^{(n)} - p^{(n)}$  vanishes at some  $\xi \in [\bar{a}, \bar{b}]$ . But  $p(x) = \frac{1}{n!} p^{(n)}(\xi) x^n + \text{lower order terms}$  (for any  $\xi \in \mathbb{R}$ ), and we deduce (6.3).  $\square$

**Application** It is a consequence of the theorem that divided differences can be used to approximate derivatives.

## 6.5 Recurrence relations for divided differences

Our next topic is a useful way to calculate divided differences (and, ultimately, to deduce yet another means to construct an interpolating polynomial). We commence with the remark that  $f[x_i]$  is the coefficient of  $x^0$  in the polynomial of degree 0 (i.e., a constant) that interpolates  $f(x_i)$ , hence  $f[x_i] = f(x_i)$ .

**Theorem** Suppose that  $x_0, x_1, \dots, x_{k+1}$  are pairwise distinct, where  $k \geq 0$ . Then

$$f[x_0, x_1, \dots, x_{k+1}] = \frac{f[x_1, x_2, \dots, x_{k+1}] - f[x_0, x_1, \dots, x_k]}{x_{k+1} - x_0}. \quad (6.4)$$

**Proof.** Let  $p, q \in \mathbb{P}_k[x]$  be the polynomials that interpolate  $f$  at

$$\{x_0, x_1, \dots, x_k\} \quad \text{and} \quad \{x_1, x_2, \dots, x_{k+1}\}$$

respectively and define

$$r(x) := \frac{(x - x_0)q(x) + (x_{k+1} - x)p(x)}{x_{k+1} - x_0} \in \mathbb{P}_{k+1}[x].$$

We readily verify that  $r(x_i) = f(x_i)$ ,  $i = 0, 1, \dots, k + 1$ . Hence  $r$  is the  $(k + 1)$ -degree interpolating polynomial and  $f[x_0, \dots, x_{k+1}]$  is the coefficient of  $x^{k+1}$  therein. The recurrence (6.4) follows from the definition of divided differences.  $\square$