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Numerical Analysis – Lecture 8

6.3 The error of polynomial interpolation

Let $[a,b]$ be a closed interval of $\mathbb{R}$. We denote by $C[a,b]$ the space of all continuous functions from $[a,b]$ to $\mathbb{R}$ and let $C^s[a,b]$, where s is a positive integer, stand for the linear space of all functions in $C[a,b]$ that possess s continuous derivatives.

Theorem Given $f \in C^{n+1}[a,b]$, let $p \in \mathbb{P}_n[x]$ interpolate the values $f(x_i)$, $i = 0,1,\ldots,n$, where $x_0,\ldots,x_n \in [a,b]$ are pairwise distinct. Then for every $x \in [a,b]$ there exists $\xi \in [a,b]$ such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i). \tag{6.1}$$

Proof. The formula (6.1) is true when $x = x_j$ for $j \in \{0,1,\ldots,n\}$, since both sides of the equation vanish. Let $x \in [a,b]$ be any other point and define

$$\phi(t) := [f(t) - p(t)] \prod_{i=0}^{n} (x - x_i) - [f(x) - p(x)] \prod_{i=0}^{n} (t - x_i), \qquad t \in [a,b].$$

[Note: The variable in ϕ is t, whereas x is a fixed parameter.] Note that $\phi(x_j) = 0$, $j = 0,1,\ldots,n$, and $\phi(x) = 0$. Hence, ϕ has at least $n+2$ distinct zeros in $[a,b]$. Moreover, $\phi \in C^{n+1}[a,b]$.
We now apply the *Rolle theorem:* if the function $g \in C^1[a,b]$ vanishes at two distinct points in $[a,b]$ then its derivative vanishes at an intermediate point. We deduce that ϕ' vanishes at (at least) $n+1$ distinct points in $[a,b]$. Next, applying Rolle to ϕ', we conclude that ϕ'' vanishes at n points in $[a,b]$. In general, we prove by induction that $\phi^{(s)}$ vanishes at $n+2-s$ distinct points of $[a,b]$ for $s = 0,1,\ldots,n+1$. Letting $s = n+1$, we have $\phi^{(n+1)}(\xi) = 0$ for some $\xi \in [a,b]$. But $p^{(n+1)} \equiv 0$, $\mathrm{d}^{n+1} \prod_{i=0}^{n} (t - x_i)/\mathrm{d}t^{n+1} \equiv (n+1)!$, and we obtain (6.1). $\square$

Runge's example We interpolate $f(x) = 1/(1+x^2)$, $x \in [-5,5]$, at the equally-spaced points $x_j = -5 + 10\frac{j}{n}$, $j = 0,1,\ldots,n$. Some of the errors for $n = 20$ are

x	$f(x) - p(x)$	$\prod_{i=0}^{n}(x - x_i)$
0.75	3.2×10^{-3}	-2.5×10^{6}
1.75	7.7×10^{-3}	-6.6×10^{6}
2.75	3.6×10^{-2}	-4.1×10^{7}
3.75	5.1×10^{-1}	-7.6×10^{8}
4.75	$4.0 \times 10^{+2}$	-7.3×10^{10}

The growth in the error is explained by the product term in (6.1) (the rightmost column of the table). Adding more interpolation points makes the largest error even worse. A remedy to this state of affairs is to cluster points toward the end of the range.

A considerably smaller error is attained for $x_j = 5\cos\frac{(n-j)\pi}{n}$, $j = 0,1,\ldots,n$ (so-called *Chebyshev points*). It is possible to prove that this choice of points minimizes the magnitude of $\max_{x\in[-5,5]} |\prod_{i=0}^{n}(x - x_i)|$.

6.4 Divided differences: a definition

Given pairwise-distinct points $x_0, x_1, \ldots, x_n \in [a,b]$, we let $p \in \mathbb{P}_n[x]$ interpolate $f \in C[a,b]$ there. The coefficient of x^n in p is called the *divided difference* and denoted by $f[x_0, x_1, \ldots, x_n]$. We say

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that this divided difference is of *degree n*.
We can derive $f[x_0, \ldots, x_n]$ from the Lagrange formula,

$$f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f(x_k) \prod_{\substack{\ell=0 \\ \ell \neq k}}^{n} \frac{1}{x_k - x_\ell}. \tag{6.2}$$

Theorem Let $[\bar{a}, \bar{b}]$ be the shortest interval that contains $x_0, x_1, \ldots, x_n$ and let $f \in C^n[\bar{a}, \bar{b}]$. Then there exists $\xi \in [\bar{a}, \bar{b}]$ such that

$$f[x_0, x_1, \ldots, x_n] = \tfrac{1}{n!} f^{(n)}(\xi). \tag{6.3}$$

Proof. Let p be the interpolating polynomial. The error function $f - p$ has at least $n+1$ zeros in $[\bar{a}, \bar{b}]$ and, applying Rolle's theorem n times, it follows that $f^{(n)} - p^{(n)}$ vanishes at some $\xi \in [\bar{a}, \bar{b}]$. But $p(x) = \frac{1}{n!} p^{(n)}(\xi) x^n +$ lower order terms (for any $\xi \in \mathbb{R}$), and we deduce (6.3). □

Application It is a consequence of the theorem that divided differences can be used to approximate derivatives.

6.5 Recurrence relations for divided differences

Our next topic is a useful way to calculate divided differences (and, ultimately, to deduce yet another means to construct an interpolating polynomial). We commence with the remark that $f[x_i]$ is the coefficient of x^0 in the polynomial of degree 0 (i.e., a constant) that interpolates $f(x_i)$, hence $f[x_i] = f(x_i)$.

Theorem Suppose that $x_0, x_1, \ldots, x_{k+1}$ are pairwise distinct, where $k \geq 0$. Then

$$f[x_0, x_1, \ldots, x_{k+1}] = \frac{f[x_1, x_2, \ldots, x_{k+1}] - f[x_0, x_1, \ldots, x_k]}{x_{k+1} - x_0}. \tag{6.4}$$

Proof. Let $p, q \in \mathbb{P}_k[x]$ be the polynomials that interpolate f at

$$\{x_0, x_1, \ldots, x_k\} \qquad \text{and} \qquad \{x_1, x_2, \ldots, x_{k+1}\}$$

respectively and define

$$r(x) := \frac{(x - x_0)q(x) + (x_{k+1} - x)p(x)}{x_{k+1} - x_0} \in \mathbb{P}_{k+1}[x].$$

We readily verify that $r(x_i) = f(x_i)$, $i = 0, 1, \ldots, k+1$. Hence r is the $(k+1)$-degree interpolating polynomial and $f[x_0, \ldots, x_{k+1}]$ is the coefficient of x^{k+1} therein. The recurrence (6.4) follows from the definition of divided differences. □

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