

Numerical Analysis – Lecture 10

7.3 The three-term recurrence relation

How to construct orthogonal polynomials? (7.1) might help, but it suffers from the same problem as the Gram–Schmidt algorithm in Euclidean spaces: loss of accuracy due to ill-conditioning. A considerably better procedure follows from our next theorem.

Theorem Monic orthogonal polynomials are given by the formula

$$\begin{aligned} p_{-1}(x) &\equiv 0, & p_0(x) &\equiv 1, \\ p_{n+1}(x) &= (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), & n &= 0, 1, \dots, \end{aligned} \quad (7.1)$$

where

$$\alpha_n := \frac{\langle p_n, xp_n \rangle}{\langle p_n, p_n \rangle}, \quad \beta_n = \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle} > 0.$$

Proof. Pick $n \geq 0$ and let

$$\psi(x) := p_{n+1}(x) - (x - \alpha_n)p_n(x) + \beta_n p_{n-1}(x).$$

Since p_n and p_{n+1} are monic, it follows that $\psi \in \mathbb{P}_n[x]$. Moreover,

$$\langle \psi, p_\ell \rangle = \langle p_{n+1}, p_\ell \rangle - \langle p_n, (x - \alpha_n)p_\ell \rangle + \beta_n \langle p_{n-1}, p_\ell \rangle = 0, \quad \ell = 0, 1, \dots, n-2.$$

Because of monicity, $xp_{n-1} = p_n + q$, where $q \in \mathbb{P}_{n-1}[x]$. Thus, from the definition of α_n, β_n ,

$$\begin{aligned} \langle \psi, p_{n-1} \rangle &= -\langle p_n, xp_{n-1} \rangle + \beta_n \langle p_{n-1}, p_{n-1} \rangle = -\langle p_n, p_n \rangle + \beta_n \langle p_{n-1}, p_{n-1} \rangle = 0, \\ \langle \psi, p_n \rangle &= -\langle xp_n, p_n \rangle + \alpha_n \langle p_n, p_n \rangle = 0. \end{aligned}$$

Every $p \in \mathbb{P}_n[x]$ that obeys $\langle p, p_\ell \rangle = 0$, $\ell = 0, 1, \dots, n$, must necessarily be the zero polynomial. For suppose that it is not so and let x^s be the highest coefficient of x in p . Then $\langle p, p_s \rangle \neq 0$. We deduce that $\psi \equiv 0$, hence (7.1) is true. \square

Example *Chebyshev polynomials* We choose the scalar product

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x) \frac{dx}{\sqrt{1-x^2}}, \quad f, g \in C[-1, 1]$$

and define $T_n \in \mathbb{P}_n[x]$ via the relation $T_n(\cos \theta) = \cos n\theta$ – hence $T_0(x) \equiv 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$ etc. Changing the integration variable,

$$\begin{aligned} \langle T_n, T_m \rangle &= \int_{-1}^1 T_n(x)T_m(x) \frac{dx}{\sqrt{1-x^2}} = \int_0^\pi \cos n\theta \cos m\theta \, d\theta \\ &= \frac{1}{2} \int_0^\pi \{\cos(n+m)\theta + \cos(n-m)\theta\} \, d\theta = 0 \end{aligned}$$

whenever $n \neq m$. The recurrence relation for Chebyshev polynomials is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

as can be verified at once from the identity

$$\cos(n+1)\theta + \cos(n-1)\theta = 2 \cos \theta \cos n\theta.$$

Note that the T_n s aren't monic, hence the inconsistency with (7.1).

7.4 Least-squares polynomial fitting

Given $f \in C[a, b]$ and a scalar product $\langle g, h \rangle = \int_a^b w(x)g(x)h(x) dx$, we wish to pick $p \in \mathbb{P}_n[x]$ so as to minimize $\langle f - p, f - p \rangle$. Again, $w(x) > 0$ for $x \in (a, b)$. Intuitively speaking, p approximates f and is an alternative to an interpolating polynomial. (The situation is similar to the one that we have already encountered in numerical linear algebra.)

Let p_0, p_1, \dots, p_n be orthogonal polynomials w.r.t. the underlying inner product, $p_\ell \in \mathbb{P}_\ell[x]$. We can represent $p = \sum_{k=0}^n c_k p_k$ for some $c_0, c_1, \dots, c_n \in \mathbb{R}$, hence, by orthogonality,

$$\langle f - p, f - p \rangle = \left\langle f - \sum_{k=0}^n c_k p_k, f - \sum_{k=0}^n c_k p_k \right\rangle = \langle f, f \rangle - 2 \sum_{k=0}^n c_k \langle p_k, f \rangle + \sum_{k=0}^n c_k^2 \langle p_k, p_k \rangle.$$

To derive optimal c_0, c_1, \dots, c_n we seek to minimize the last expression. Since

$$\frac{1}{2} \frac{\partial}{\partial c_k} \langle f - p, f - p \rangle = -\langle p_k, f \rangle + c_k \langle p_k, p_k \rangle, \quad k = 0, 1, \dots, n,$$

setting the gradient to zero yields

$$p(x) = \sum_{k=0}^n \frac{\langle p_k, f \rangle}{\langle p_k, p_k \rangle} p_k(x). \quad (7.2)$$

Note that

$$\langle f - p, f - p \rangle = \langle f, f \rangle - \sum_{k=0}^n \{2c_k \langle p_k, f \rangle - c_k^2 \langle p_k, p_k \rangle\} = \langle f, f \rangle - \sum_{k=0}^n \frac{\langle p_k, f \rangle^2}{\langle p_k, p_k \rangle}. \quad (7.3)$$

How to choose n ? Note that $c_k = \langle p_k, f \rangle / \langle p_k, p_k \rangle$ is independent of n . Thus, we can continue to add terms to (7.2) until $\langle f - p, f - p \rangle$ is below specified *tolerance* ε . Because of (7.3), we need to pick n so that $\langle f, f \rangle - \varepsilon < \sum_{k=0}^n \langle p_k, f \rangle^2 / \langle p_k, p_k \rangle$.

Theorem (*The Parseval identity*) Let $[a, b]$ be finite. Then

$$\sum_{k=0}^{\infty} \frac{\langle p_k, f \rangle^2}{\langle p_k, p_k \rangle} = \langle f, f \rangle. \quad (7.4)$$

Incomplete proof. Let $\sigma_n := \sum_{k=0}^n \langle p_k, f \rangle^2 / \langle p_k, p_k \rangle$, $n = 0, 1, \dots$, hence $\langle f - p, f - p \rangle = \langle f, f \rangle - \sigma_n \geq 0$. The sequence $\{\sigma_n\}_{n=0}^{\infty}$ increases monotonically and $\sigma_n \leq \langle f, f \rangle$ implies that $\lim_{n \rightarrow \infty} \sigma_n$ exists. According to the *Weierstrass theorem*, any function in $C[a, b]$ can be approximated arbitrarily close by a polynomial, hence $\lim_{n \rightarrow \infty} \langle f - p, f - p \rangle = 0$ and we deduce that $\sigma_n \xrightarrow{n \rightarrow \infty} \langle f, f \rangle$ and (7.4) is true. \square