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Numerical Analysis – Lecture 12

8 The Peano kernel theorem

8.1 The theorem

Our point of departure is the *Taylor formula with an integral remainder term,*

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^k}{k!}f^{(k)}(a)+\frac{1}{k!}\int_a^x (x-\theta)^k f^{(k+1)}(\theta)\,\mathrm{d}\theta,\quad (8.1)$$

which can be verified by integration by parts. Suppose that we are given an approximant (e.g. to a function, a derivative, an integral etc.) whose error vanishes for all $f\in\mathbb{P}_k[x]$. The Taylor formula produces an expression for the error that depends on $f^{(k+1)}$. This is the basis for the *Peano kernel theorem.*

Formally, let $L(f)$ be an error of an approximant. Thus, L maps $C[a,b]$, say, to $\mathbb{R}$. We assume that it is *linear*, i.e. $L(\alpha f+\beta g)=\alpha L(f)+\beta L(g)\ \forall\alpha,\beta\in\mathbb{R}$, and that $L(f)=0$ for all $f\in\mathbb{P}_k[x]$. Thus, (8.1) implies

$$L(f)=\frac{1}{k!}L\left\{\int_a^x (x-\theta)^k f^{(k+1)}(\theta)\,\mathrm{d}\theta\right\},\qquad a\le x\le b.$$

To make the range of integration independent of x, we introduce the notation

$$(x-\theta)_+^k:=\begin{cases}(x-\theta)^k, & x\ge\theta,\\ 0, & x\le\theta,\end{cases}\qquad\text{whence}\qquad L(f)=\frac{1}{k!}L\left\{\int_a^b (x-\theta)_+^k f^{(k+1)}(\theta)\,\mathrm{d}\theta\right\}.$$

Let $K(\theta):=L[(x-\theta)_+^k]$ for $x\in[a,b]$. *[Note: K is independent of f.]* Suppose that it is allowed to exchange the order of action of $\int$ and L. Because of the linearity of L, we then have

$$L(f)=\frac{1}{k!}\int_a^b K(\theta)f^{(k+1)}(\theta)\,\mathrm{d}\theta.\qquad (8.2)$$

The Peano kernel theorem Let L be a *linear functional* (a linear mapping from a space of functions to $\mathbb{R}$) such that $L(f)=0$ for all $f\in\mathbb{P}_k[x]$. Provided that $f\in C^{k+1}[a,b]$ and the above exchange of L with the integration sign is valid, the formula (8.2) is true. □

8.2 An example and few useful formulae

Let $L(f):=f'(0)-[-\frac{3}{2}f(0)+2f(1)-\frac{1}{2}f(2)]$ – this corresponds to approximating

$$f'(0)\approx-\frac{3}{2}f(0)+2f(1)-\frac{1}{2}f(2).$$

Then $L(f)=0$ for $f\in\mathbb{P}_2[x]$ (verify by trying $f(x)=1,x,x^2$ and invoking linearity). Thus, for $f\in C^3[0,2]$ we have

$$L(f)=\frac{1}{2}\int_0^2 K(\theta)f'''(\theta)\,\mathrm{d}\theta.$$

To evaluate the *Peano kernel* K, we fix θ. Letting $g(x) := (x-\theta)_+^2$, we have

$$\begin{aligned} K(\theta) &= L(g) = g'(0) - \left[-\tfrac{3}{2}g(0) + 2g(1) - \tfrac{1}{2}g(2)\right] \\ &= 2(0-\theta)_+ - \left[-\tfrac{3}{2}(0-\theta)_+^2 + 2(1-\theta)_+^2 - \tfrac{1}{2}(2-\theta)_+^2\right] \\ &= \begin{cases} -2\theta + \frac{3}{2}\theta^2 + (2\theta - \frac{3}{2}\theta^2) = 0, & \theta \le 0, \\ -2(1-\theta)^2 + \frac{1}{2}(2-\theta)^2 = 2\theta - \frac{3}{2}\theta^2, & 0 \le \theta \le 1, \\ \frac{1}{2}(2-\theta)^2, & 1 \le \theta \le 2, \\ 0, & \theta \ge 2. \end{cases} \end{aligned}$$

[Note: It is obvious that $K(\theta) = 0$ for $\theta \notin [0,2]$, since then it acts on a quadratic polynomial.]

Back to the general case. . . Typically, L involves differentiation and integration. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(x-\theta)_+^k = k(x-\theta)_+^{k-1}, \qquad \int_0^x (t-\theta)_+^k \,\mathrm{d}t = \frac{1}{k+1}[(x-\theta)_+^{k+1} - (a-\theta)_+^{k+1}],$$

the exchange of L with integration is justified in these cases.

Theorem Suppose that K doesn't change sign in (a,b) and that $f \in C^{k+1}[a,b]$. Then

$$L(f) = \frac{1}{k!}\left[\int_a^b K(\theta)\,\mathrm{d}\theta\right] f^{(k+1)}(\xi) \quad \text{for some} \quad \xi \in (a,b).$$

Proof. Let (perversely!) $K \le 0$. Then

$$L(f) \le \frac{1}{k!}\int_a^b K(\theta) \min_{x\in[a,b]} f^{(k+1)}(x)\,\mathrm{d}\theta = \frac{1}{k!}\left(\int_a^b K(\theta)\,\mathrm{d}\theta\right) \min_{x\in[a,b]} f^{(k+1)}(x).$$

Likewise $L(f) \ge \frac{1}{k!}\left[\int_a^b K(\theta)\,\mathrm{d}\theta\right] \max_{x\in[a,b]} f^{(k+1)}(x)$, consequently

$$\min_{x\in[a,b]} f^{(k+1)}(x) \le \frac{L[f]}{\frac{1}{k!}\int_a^b K(\theta)\,\mathrm{d}\theta} \le \max_{x\in[a,b]} f^{(k+1)}(x)$$

and the required result follows from the mean value theorem. Similar analysis pertains to the case $K \ge 0$. □

Back to our example We have $K \ge 0$ and $\int_0^2 K(\theta)\,\mathrm{d}\theta = \frac{2}{3}$. Consequently $L(f) = \frac{1}{2!} \times \frac{2}{3} f'''(\xi) = \frac{1}{3}f'''(\xi)$ for some $\xi \in (0,2)$. We deduce in particular that $|L(f)| \le \frac{1}{3}\|f'''\|_\infty$, where $\|g\|_\infty := \max_{x\in[0,2]} |g(x)|$ – the ∞-*norm.*

Likewise, generalising the definition of the ∞-norm to an arbitrary interval $[a,b]$, we can easily deduce from

$$\left|\int_a^b f(x)g(x)\,\mathrm{d}x\right| \le \|g\|_\infty \int_a^b |f(x)|\,\mathrm{d}x,$$

that $|L(f)| \le \frac{1}{k!}\int_a^b |K(\theta)|\,\mathrm{d}\theta \|f^{(k+1)}\|_\infty$ and that $|L(f)| \le \frac{1}{k!}\|K\|_\infty \int_a^b |f^{(k+1)}(x)|\,\mathrm{d}x$. This is valid also when K changes sign. Moreover, letting $\|f\|_2 := \left[\int_a^b |f(x)|^2\,\mathrm{d}x\right]^{1/2}$ – the 2-*norm* – the *Cauchy–Schwarz inequality* $\left|\int_a^b f(x)g(x)\,\mathrm{d}x\right| \le \|f\|_2\|g\|_2$ implies that $|L(f)| \le \frac{1}{k!}\|K\|_2\|f^{(k+1)}\|_2$.