A. Shadrin

Mathematical Tripos Part IB: Lent Term 2024
Numerical Analysis – Examples' Sheet 1

1. Suppose that the function values $f(0), f(1), f(2)$ and $f(3)$ are given and that we wish to estimate

$$f(6), \quad f'(0) \quad \text{and} \quad \int_0^3 f(x)\,dx.$$

One method is to let p be the cubic polynomial that interpolates these function values, and then to employ the approximants

$$p(6), \quad p'(0) \quad \text{and} \quad \int_0^3 p(x)\,dx$$

respectively. Deduce from the Lagrange formula for p that each approximant is a linear combination of the four data with constant coefficients. Calculate the numerical values of these constants. Verify your work by showing that the approximants are exact when f is an arbitrary cubic polynomial.

2. Let f be a function in $C^4[0,1]$ and let p be a cubic polynomial that interpolates $f(0), f'(0), f(1)$ and $f'(1)$. Deduce from the Rolle theorem that for every $x \in [0,1]$ there exists $\xi \in [0,1]$ such that the equation

$$f(x) - p(x) = \tfrac{1}{24} x^2 (x-1)^2 f^{(4)}(\xi)$$

is satisfied.

3. Let a, b and c be distinct real numbers (not necessarily in ascending order), and let $f(a)$, $f(b)$, $f'(a)$, $f'(b)$ and $f'(c)$ be given. Because there are five data, one might try to approximate f by a polynomial of degree at most four that interpolates the data. Prove by a general argument that this interpolation problem has a solution and the solution is unique if and only if there is no nonzero polynomial $p \in \mathcal{P}_4$ that satisfies $p(a) = p(b) = p'(a) = p'(b) = p'(c) = 0$. Hence, given a and b, show that there exists a unique value of $c \neq a, b$ such that there is no unique solution.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function and let p be the polynomial of degree at most n that interpolates f at the pairwise distinct points $x_0, x_1, \ldots, x_n$. Further, let x be any real number that is not an interpolation point. Deduce the identity

$$f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

from the definition of the divided difference $f[x_0, x_1, \ldots, x_n, x]$.

5. Simulating a computer that works to only four decimal places, form the table of divided differences of the values $f(0) = 0$, $f(0.1) = 0.0998$, $f(0.4) = 0.3894$ and $f(0.7) = 0.6442$ of $\sin x$. Hence identify the polynomial that is given by Newton's interpolation method. Due to rounding errors, this polynomial should differ from the one that would be given by exact arithmetic. Take the view, however, that the *computed* values of $f[0.0, 0.1]$, $f[0.0, 0.1, 0.4]$ and $f[0.0, 0.1, 0.4, 0.7]$ and the function value $f(0)$ are correct. Then, by working backwards through the difference table, identify the values of $f(0), f(0.1), f(0.4)$ and $f(0.7)$ that would give these divided differences in exact arithmetic.

6. Set $f(x) = 2x - 1$, $x \in [0, 1]$. We require a function of form

$$p(x) = \sum_{k=0}^{n} a_k \cos(k\pi x), \quad 0 \le x \le 1,$$

that satisfies the condition

$$\int_0^1 [f(x) - p(x)]^2 \, dx < 10^{-4}.$$

Explain why it is sufficient if the value of $a_0^2 + \frac{1}{2}\sum_{k=1}^{n} a_k^2$ exceeds $\frac{1}{3} - 10^{-4}$, where the coefficients $\{a_k\}_{k=0}^{n}$ are calculated to minimize this integral. Hence find the smallest acceptable value of n.

7. The polynomials $\{p_n\}_{n \in \mathbb{Z}^+}$ are defined by the three-term recurrence formula

$$\begin{aligned} p_0(x) &\equiv 1, \\ p_1(x) &= 2x, \\ p_{n+1}(x) &= 2xp_n(x) - p_{n-1}(x), \qquad n = 1, 2, \ldots. \end{aligned}$$

Prove that they are orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\sqrt{1 - x^2} \, dx$$

and evaluate (p_n, p_n) for $n \in \mathbb{Z}^+$. [*Hint*: Prove that $p_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, where $x = \cos\theta$.]

8. Calculate the weights a_1, a_2 and nodes x_1, x_2 so that the approximant

$$\int_0^1 f(x) \, dx \approx a_1 f(x_1) + a_2 f(x_2)$$

is exact when f is a cubic polynomial. You may exploit the fact that x_1 and x_2 are the zeros of a quadratic polynomial that is orthogonal to all linear polynomials. Verify your calculation by testing the formula when $f(x) = 1, x, x^2$ and x^3.

9. The functions $p_0, p_1, p_2, \ldots$ are generated by the Rodrigues formula

$$p_n(x) = \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \mathrm{e}^{-x}), \qquad 0 \le x < \infty.$$

Show that these functions are polynomials and prove by integration by parts that for every $p \in \mathcal{P}_{n-1}$ we have the orthogonality condition $(p_n, p) = 0$ with respect to the scalar product

$$(f, g) := \int_0^\infty f(x)g(x)\,\mathrm{e}^{-x} dx.$$

Derive the coefficients of p_3, p_4 and p_5 from the Rodrigues formula. Verify that these coefficients are compatible with a three term recurrence relation of the form

$$p_5(x) = (\gamma x - \alpha)p_4(x) - \beta p_3(x), \qquad x \in \mathbb{R},$$

where α, β and γ are constants.

10. Let $p(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))$, where f is a function in $C^2[0, 1]$. Find the least constants c_0, c_1 and c_2 such that the error bounds

$$\left|f(\tfrac{1}{2}) - p(\tfrac{1}{2})\right| \le c_k \|f^{(k)}\|_\infty, \qquad k = 0, 1, 2,$$

are valid. [*Note*: The cases $k = 0$ and $k = 1$ are easy if one works from first principles, and the Peano kernel theorem is suitable when $k = 2$. Also try the Peano kernel theorem when $k = 1$.]

11. Express the divided difference $f[0,1,2,4]$ in the form

$$f[0,1,2,4] = \frac{1}{2}\int_0^4 K(t) f'''(t)\, dt,$$

assuming that f''' exists and is continuous. Sketch the kernel function $K(t)$ for $0 \leq t \leq 4$. By integrating $K(t)$ analytically and using the integral mean value theorem prove that

$$f[0,1,2,4] = \tfrac{1}{6} f'''(\xi)$$

for some point $\xi \in [0,4]$. Note that another proof of this result was given in the lecture on divided differences. (The integral mean value theorem: if $f(t) \geq 0$ on $[a,b]$ and $g \in C[a,b]$, then $\int_a^b f(t)g(t)\, dt = g(\xi)\int_a^b f(t)\, dt$ for some $\xi \in [a,b]$.)

12*. Let f be a function in $C^4[0,1]$ and let ξ be any fixed point in $[0,1]$. Calculate the coefficients α, β, γ and δ such that the approximant

$$f'''(\xi) \approx \alpha f(0) + \beta f(1) + \gamma f'(0) + \delta f'(1)$$

is exact for all cubic polynomials. Prove that the inequality

$$\left| f'''(\xi) - \alpha f(0) - \beta f(1) - \gamma f'(0) - \delta f'(1) \right| \leq \left\{ \tfrac{1}{2} - \xi + 2\xi^3 - \xi^4 \right\} \| f^{(4)} \|_\infty$$

is satisfied. Show that this inequality holds as an equation if we allow f to be the function

$$f(x) = \begin{cases} -(x-\xi)^4, & 0 \leq x \leq \xi, \\ (x-\xi)^4, & \xi \leq x \leq 1. \end{cases}$$

13*. Given f and g in $C[a,b]$, let $h := fg$. Prove by induction that the divided differences of h satisfy the equation

$$h[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k]\, g[x_k, x_{k+1}, \ldots, x_n].$$

By expressing the differences in terms of derivatives and by letting the points $x_0, x_1, \ldots, x_n$ become coincident, deduce the Leibnitz formula for the nth derivative of a product of two functions.