1. Given $h > 0$, let Euler’s method be applied to calculate the estimates $\{y_n\}$ of $y(nh)$ for each of the differential equations

$$y' = -\frac{y}{1 + t} \quad \text{and} \quad y' = \frac{2y}{1 + t}, \quad 0 \leq t \leq 1,$$

starting with $y_0 = y(0) = 1$ in both cases. By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for $y_n$, which should be free from summations and products of $n$ terms. Hence deduce the exact solutions of the equations from the limit $h \to 0$ and $nh \to t$. Verify that, for $nh \in [0, 1]$, the magnitude of the errors $y_n - y(nh)$ is at most $O(h)$.

2. Assuming that $f$ satisfies the Lipschitz condition and possesses a bounded third derivative in $[0, T]$, apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

converges and that $\|y_n - y(t_n)\| \leq ch^2$ for some $c > 0$ and all $n$ such that $0 \leq nh \leq T$.

3. The $s$-step Adams–Bashforth method is of order $s$ and has the form

$$y_{n+s} = y_{n+s-1} + h \sum_{m=0}^{s-1} b_m f(t_{n+m}, y_{n+m}).$$

Calculate the actual values of the coefficients in the case $s = 3$.

4. By solving a three-term recurrence relation, calculate analytically the sequence of values $\{y_n\}_{n \geq 2}$ that is generated by the explicit midpoint rule

$$y_{n+2} = y_n + 2hf(t_{n+1}, y_{n+1}),$$

when it is applied to the ODE $y' = -y$, $t \geq 0$. Starting from the values $y_0 = 1$ and $y_1 = 1 - h$, show that the sequence diverges as $n \to \infty$ for any $h > 0$. Recall, however, that order $\geq 1$, the root condition and suitable starting conditions imply convergence in a finite interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem.

[Hint. In the last part, relate the roots of the recurrence relation to $\pm e^{\pm h} + O(h^2)$.]}

5. Show that the multistep method (with $a_3 = 1$)

$$\sum_{m=0}^{3} a_m y_{n+m} = h \sum_{m=0}^{2} b_m f(t_{n+m}, y_{n+m})$$

is fourth order only if the conditions $a_0 + a_2 = 8$ and $a_1 = -9$ are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition. (Compare this conclusion with Theorem 8.9.)

6. An $s$-stage explicit Runge–Kutta method with constant step size $h > 0$ is applied to the differential equation $y' = \lambda y$, $t \geq 0$. Prove that, with some polynomial $p_s$ of degree $s$, we have

$$y_{n+1} = p_s(\lambda h)y_n, \quad p_s \in P_s.$$

Hence derive that no explicit Runge-Kutta method can be $A$-stable (or even $A_0$-stable). Further, prove that if an $s$-stage RK method is of order $s$, then

$$p_s(\lambda h) = \sum_{m=0}^{s} \frac{1}{m!} (\lambda h)^m.$$
7. The following four-stage Runge–Kutta method has order four,

\[ \begin{align*}
    k_1 &= f(t_n, y_n) \\
    k_2 &= f(t_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1) \\
    k_3 &= f(t_n + \frac{2}{3}h, y_n - \frac{1}{3}hk_1 + hk_2) \\
    k_4 &= f(t_n + h, y_n + hk_1 - hk_2 + hk_3) \\
    y_{n+1} &= y_n + h \left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right).
\end{align*} \]

By considering the equation \( y' = y \), show that the order is at most four. Then, for scalar \( f \), prove that the order is at least four in the case when \( f \) is independent of \( y \). (In the last case, you may relate the scheme to a quadrature formula.)

8. Find \( D \cap \mathbb{R} \), the intersection of the linear stability domain \( D \) with the real axis, for the following methods:

\[ \begin{align*}
    (1) \quad & y_{n+1} = y_n + hf(t_n, y_n) \\
    (2) \quad & y_{n+1} = y_n + \frac{1}{2}h[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \\
    (3) \quad & y_{n+2} = y_n + 2hf(t_{n+1}, y_{n+1}) \\
    (4) \quad & y_{n+2} = y_n + \frac{1}{2}h[3f(t_{n+1}, y_{n+1}) - f(t_n, y_n)] \\
    (5) \quad & \text{The RK method: } k_1 = f(t_n, y_n), \quad k_2 = f(t_n + h, y_n + hk_1), \quad y_{n+1} = y_n + \frac{1}{2}h(k_1 + k_2).
\end{align*} \]

9. Show that, if \( z \) is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method

\[ y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(t_{n+2}, y_{n+2}), \]

then the real part of \( z \) is positive. Thus deduce that this method is A-stable.

10. The (stiff) differential equation

\[ y'(t) = -10^4(y - t^{-1}) - t^{-2}, \quad t \geq 1, \quad y(1) = 1, \]

has the analytic solution \( y(t) = t^{-1}, t \geq 1 \). Let it be solved numerically by Euler’s method \( y_{n+1} = y_n + hf(t_n, y_n) \) and the backward Euler method \( y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \), where \( h_n = t_{n+1} - t_n \) is allowed to depend on \( n \) and to be different in the two cases. Suppose that, for any \( t_n \geq 1 \), we have \( |y_n - y(t_n)| \leq 10^{-6} \), and that we require \( |y_{n+1} - y(t_{n+1})| \leq 10^{-6} \). Show that Euler’s method can fail if \( h_n = 2 \times 10^{-4} \), but that the backward Euler method always succeeds if \( h_n \leq 10^{-2}t_n^2 \).

Hint: Find relations between \( y_{n+1} - y(t_{n+1}) \) and \( y_n - y(t_n) \) for general \( y_n \) and \( t_n \).

11. This question concerns the predictor-corrector pair

\[ \begin{align*}
    y_{n+3}^P &= -\frac{1}{2}y_n + 3y_{n+1} - \frac{3}{2}y_{n+2} + 3hf(t_{n+2}, y_{n+2}), \\
    y_{n+3}^C &= \frac{1}{11}[2y_n - 9y_{n+1} + 18y_{n+2} + 6hf(t_{n+3}, y_{n+3})].
\end{align*} \]

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne’s device has the value \( \frac{\alpha}{17}[y_{n+3}^P - y_{n+3}^C] \).

12. Let \( u(x), 0 \leq x \leq 1, \) be a six-times differentiable function that satisfies the ODE \( u'' = f(x), 0 \leq x \leq 1, u(0) \) and \( u(1) \) being given. Further, we let \( x_m = mh = m/M, m = 0, 1, \ldots, M, \) for some positive integer \( M, \) and calculate the estimates \( u_m \approx u(x_m), m = 1, 2, \ldots, M - 1, \) by solving the difference equation

\[ u_{m-1} - 2u_m + u_{m+1} = h^2f(x_m) + \alpha h^2[f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \quad m = 1, 2, \ldots, M - 1, \]

where \( u_0 = u(0), u_M = u(1), \) and \( \alpha \) is a positive parameter. Show that there exists a choice of \( \alpha \) such that the local truncation error of the difference equation is \( O(h^6) \).