Numerical Analysis: Example Sheet 2

1. Let \( h = 1/M \), where \( M \geq 1 \) is an integer. Consider the differential equations

\[
y'(t) = -\frac{y}{1 + t} \quad \text{and} \quad y'(t) = \frac{2y}{1 + t}, \quad 0 \leq t \leq 1,
\]

with initial condition \( y(0) = 1 \) in both cases. In each case find the exact solution of the differential equation.

Define Euler’s method, and apply it to calculate the estimates \( \{y_n\}_{n=1}^{M} \) of \( y(nh) \) for each of the equations letting \( y_0 = y(0) = 1 \). By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for \( y_n, n = 1, 2, \ldots, M \), which should be free from summations and products of \( n \) terms. Deduce the exact solutions of the equations from the limit \( h \to 0 \). Verify that the magnitude of the errors \( y_n - y(nh), n = 1, 2, \ldots, M \), is at most \( O(h) \).

2. Assuming that \( f \) satisfies the Lipschitz condition and the true solution possesses a bounded third derivative in \([0, t^*]\), apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

\[
y_{n+1} = y_n + \frac{1}{2}h[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]
\]

converges and that \( \|y_n - y(t_n)\| \leq ch^2 \) for some \( c > 0 \) and all \( n \) such that \( 0 \leq nh \leq t^* \).

3. The \( s \)-step Adams–Bashforth method is of order \( s \) and has the form

\[
y_{n+s} = y_{n+s-1} + \frac{s-1}{2} \sum_{j=0}^{s-1} \sigma_j f(t_{n+j}, y_{n+j}).
\]

Calculate the actual values of the coefficients in the case \( s = 3 \).

4. By solving a three-term recurrence relation, calculate analytically the sequence of values \( \{y_n : n = 2, 3, 4, \ldots\} \) that is generated by the explicit midpoint rule

\[
y_{n+2} = y_n + 2hf(t_{n+1}, y_{n+1}),
\]

when it is applied to the ODE \( y' = -y, t \geq 0 \). Starting from the values \( y_0 = 1 \) and \( y_1 = 1 - h \), show that the sequence diverges as \( n \to \infty \) for all \( h > 0 \).

However if the order of the method is greater or equal to one, the root condition and suitable starting conditions imply convergence in a finite interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem. 

Hint: in the last part, relate the roots of the recurrence relation to \( \pm e^{\mp n} + O(h^3) \).

5. Show that the multistep method

\[
\sum_{j=0}^{3} \rho_j y_{n+j} = h \sum_{j=0}^{2} \sigma_j f(t_{n+j}, y_{n+j}), \quad \text{where} \quad \rho_0 = 1 \quad \text{(as in lectures),}
\]

is fourth order only if the conditions \( \rho_0 + \rho_2 = 8 \) and \( \rho_1 = -9 \) are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition.

6. An \( s \)-stage explicit Runge–Kutta method of order \( s \) with constant step size \( h > 0 \) is applied to the differential equation \( y' = \lambda y, t \geq 0 \). Prove the identity

\[
y_n = \left[ \sum_{\ell=0}^{s} \frac{1}{\ell!} (\lambda h)^\ell \right]^n y_0, \quad n = 0, 1, 2, \ldots
\]

by showing by induction that \( k_i = \lambda y_p p_{i-1}(\lambda h) \) where the \( p_{i-1} \) are polynomials of degree \( i - 1 \), deducing that \( y_{n+1} = p_s(\lambda h)y_n \), and using the order condition to determine \( p_s \); or otherwise.
7. The following four-stage Runge–Kutta method has order four,
\[ k_1 = f(t_n, y_n) \]
\[ k_2 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \]
\[ k_3 = f(t_n + \frac{1}{2}h, y_n - \frac{1}{2}hk_1 + hk_2) \]
\[ k_4 = f(t_n + h, y_n + hk_1 - hk_2 + 2hk_3) \]
\[ y_{n+1} = y_n + h(\frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4). \]

By considering the equation \( y' = y \), show that the order is at most four. Then, for scalar functions, prove that the order is at least four in the easy case when \( f \) is independent of \( y \), and that the order is at least three in the relatively easy case when \( f \) is independent of \( t \).

**Comment:** do not derive all of the (gory) details when \( f(t, y) \) depends on both \( t \) and \( y \).

8. Find \( \mathcal{D} \cap \mathbb{R} \), the intersection of the linear stability domain \( \mathcal{D} \) with the real axis, for the following methods:

- (1) \( y_{n+1} = y_n + h f(t_n, y_n) \)
- (2) \( y_{n+1} = y_n + \frac{1}{2}h[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \)
- (3) \( y_{n+2} = y_n + 2hf(t_{n+1}, y_{n+1}) \)
- (4) \( y_{n+2} = y_{n+1} + \frac{1}{2}h[3f(t_{n+1}, y_{n+1}) - f(t_n, y_n)] \)
- (5) The RK method \( k_1 = f(t_n, y_n), k_2 = f(t_n + h, y_n + hk_1), y_{n+1} = y_n + \frac{1}{2}h(k_1 + k_2). \)

*Hint:* note that to solve \( a < X < b \), instead of manipulating the inequalities it can be easier to solve \( X = a \) and \( X = b \), and then decide which region is required by considering one interior point (as for conformal maps).

9. Show that, if \( z \) is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method
\[ y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(t_{n+2}, y_{n+2}) \]
then the real part of \( z \) is positive. Thus deduce that this method is A-stable.

*Hint:* if \( z \) is on the boundary of the linear stability domain then \( y_n = \exp(i\theta) \) for some real \( \theta \).

10. The (stiff) differential equation
\[ y'(t) = -10^4(y - t^{-1}) - t^{-2}, \quad t \geq 1, \quad y(1) = 1, \]
has the analytic solution \( y(t) = t^{-1}, \ t \geq 1 \). Let it be solved numerically by Euler’s method \( y_{n+1} = y_n + h f(t_n, y_n) \) and the backward Euler method \( y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) \), where \( h_n = t_{n+1} - t_n \) is allowed to depend on \( n \) and to be different in the two cases. Suppose that, for any \( t_n \geq 1 \), we have \( |y_n - y(t_n)| \leq 10^{-6} \), and that we require \( |y_{n+1} - y(t_{n+1})| \leq 10^{-6} \). Show that Euler’s method can fail if \( h_n = 2 \times 10^{-4} \), but that the backward Euler method always succeeds if \( h_n \leq 10^{-2}t_n^2/101 \).

*Hint:* find relations between \( y_{n+1} - y(t_{n+1}) \) and \( y_n - y(t_n) \) for general \( y_n \) and \( t_n \).

11. This question concerns the predictor-corrector pair
\[ y_{n+3}^P = -\frac{1}{2}y_n + 3y_{n+1} - \frac{3}{2}y_{n+2} + 3hf(t_{n+2}, y_{n+2}), \]
\[ y_{n+3}^C = \frac{1}{11}[2y_n - 9y_{n+1} + 18y_{n+2} + 6hf(t_{n+3}, y_{n+3})]. \]

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne’s device has the value \( \frac{1}{27}y_{n+3}^C - y_{n+3}^C \).

12. Let \( u(x), 0 \leq x \leq 1, \) be a six-times differentiable function that satisfies the ODE \( u''(x) = f(x) \), \( 0 \leq x \leq 1, u(0) \) and \( u(1) \) being given. Further, we let \( x_m = mh = m/M, m = 0, 1, \ldots, M \), for some positive integer \( M \), and calculate the estimates \( u_m \approx u(x_m), m = 1, 2, \ldots, M - 1 \), by solving the difference equation
\[ u_{m+1} - 2u_m + u_{m-1} = h^2 f(x_m) + \alpha h^2[f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \quad m = 1, 2, \ldots, M - 1, \]
where \( u_0 = u(0), u_M = u(1) \), and \( \alpha \) is a positive parameter. Show that there exists a choice of \( \alpha \) such that the local truncation error of the difference equation is \( O(h^6) \).