

B13b

## Numerical Analysis: Example Sheet 2

Lent 2011

A \* denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** necessarily harder than unstarred questions.

Corrections and suggestions should be emailed to [S.J.Cowley@damtp.cam.ac.uk](mailto:S.J.Cowley@damtp.cam.ac.uk).

1. Let  $h = 1/M$ , where  $M \geq 1$  is an integer. Consider the differential equations

$$y' = -\frac{y}{1+t} \quad \text{and} \quad y' = \frac{2y}{1+t}, \quad 0 \leq t \leq 1, \quad (*)$$

with initial condition  $y(0) = 1$  in both cases. In each case find the exact solution of the differential equation.

Define Euler's method, and apply it to calculate the estimates  $\{\mathbf{y}_n\}_{n=1,2,\dots,M}$  of  $\mathbf{y}(nh)$  for each of the equations letting  $y_0 = y(0) = 1$ . By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for  $y_n$ ,  $n = 1, 2, \dots, M$ , which should be free from summations and products of  $n$  terms. Deduce the exact solutions of the equations from the limit  $h \rightarrow 0$ . Verify that the magnitude of the errors  $y_n - y(nh)$ ,  $n = 1, 2, \dots, M$ , is at most  $\mathcal{O}(h)$ .

2. Assuming that  $\mathbf{f}$  satisfies the Lipschitz condition and the true solution possesses a bounded third derivative in  $[0, t^*]$ , apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]$$

converges and that  $\|\mathbf{y}_n - \mathbf{y}(t_n)\| \leq ch^2$  for some  $c > 0$  and all  $n$  such that  $0 \leq nh \leq t^*$ .

3. The  $s$ -step Adams–Bashforth method is of order  $s$  and has the form

$$\mathbf{y}_{n+s} = \mathbf{y}_{n+s-1} + h \sum_{j=0}^{s-1} \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j}).$$

Calculate the actual values of the coefficients in the case  $s = 3$

4. By solving a three-term recurrence relation, calculate analytically the sequence of values  $\{\mathbf{y}_n : n = 2, 3, 4, \dots\}$  that is generated by the *explicit midpoint rule*

$$\mathbf{y}_{n+2} = \mathbf{y}_n + 2h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}),$$

when it is applied to the ODE  $y' = -y$ ,  $t \geq 0$ . Starting from the values  $y_0 = 1$  and  $y_1 = 1 - h$ , show that the sequence diverges as  $n \rightarrow \infty$  for *all*  $h > 0$ .

However if the order of the method is greater or equal to one, the root condition and suitable starting conditions imply convergence in a *finite* interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem.

*Hint:* in the last part, relate the roots of the recurrence relation to  $\pm e^{\pm h} + \mathcal{O}(h^3)$ .

5. Show that the multistep method

$$\sum_{j=0}^3 \rho_j \mathbf{y}_{n+j} = h \sum_{j=0}^2 \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j}), \quad \text{where } \rho_3 = 1 \text{ (as in lectures),}$$

is fourth order only if the conditions  $\rho_0 + \rho_2 = 8$  and  $\rho_1 = -9$  are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition

- \*6. An  $s$ -stage explicit Runge–Kutta method of order  $s$  with constant step size  $h > 0$  is applied to the differential equation  $y' = \lambda y$ ,  $t \geq 0$ . Prove the identity

$$y_n = \left[ \sum_{\ell=0}^s \frac{1}{\ell!} (h\lambda)^\ell \right]^n y_0, \quad n = 0, 1, 2, \dots,$$

by showing by induction that  $k_i = \lambda y_n p_{i-1}(\lambda h)$  where the  $p_{i-1}$  are polynomials of degree  $i - 1$ , deducing that  $y_{n+1} = p_s(\lambda h)y_n$ , and using the order condition to determine  $p_s$ ; or otherwise.

7. The following four-stage Runge–Kutta method has order four,

$$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$$

$$\mathbf{k}_2 = \mathbf{f}(t_n + \frac{1}{3}h, \mathbf{y}_n + \frac{1}{3}h\mathbf{k}_1)$$

$$\mathbf{k}_3 = \mathbf{f}(t_n + \frac{2}{3}h, \mathbf{y}_n - \frac{1}{3}h\mathbf{k}_1 + h\mathbf{k}_2)$$

$$\mathbf{k}_4 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_1 - h\mathbf{k}_2 + h\mathbf{k}_3)$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h(\frac{1}{8}\mathbf{k}_1 + \frac{3}{8}\mathbf{k}_2 + \frac{3}{8}\mathbf{k}_3 + \frac{1}{8}\mathbf{k}_4).$$

By considering the equation  $y' = y$ , show that the order is at most four. Then, for scalar functions, prove that the order is at least four in the easy case when  $f$  is independent of  $y$ , and that the order is at least three in the relatively easy case when  $f$  is independent of  $t$ .

*Comment:* do not derive all of the (gory) details when  $f(t, y)$  depends on both  $t$  and  $\mathbf{y}$ .

8. Find  $\mathcal{D} \cap \mathbb{R}$ , the intersection of the linear stability domain  $\mathcal{D}$  with the real axis, for the following methods:

$$(1) \mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_n, \mathbf{y}_n) \quad (2) \mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]$$

$$(3) \mathbf{y}_{n+2} = \mathbf{y}_n + 2h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) \quad (4) \mathbf{y}_{n+2} = \mathbf{y}_{n+1} + \frac{1}{2}h[3\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - \mathbf{f}(t_n, \mathbf{y}_n)]$$

$$(5) \text{ The RK method } \mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n), \quad \mathbf{k}_2 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_1), \quad \mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h(\mathbf{k}_1 + \mathbf{k}_2).$$

*Hint:* note that to solve  $a < X < b$ , instead of manipulating the inequalities it can be easier to solve  $X = a$  and  $X = b$ , and then decide which region is required by considering one interior point (as for conformal maps).

9. Show that, if  $z$  is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method

$$\mathbf{y}_{n+2} - \frac{4}{3}\mathbf{y}_{n+1} + \frac{1}{3}\mathbf{y}_n = \frac{2}{3}h\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})$$

then the real part of  $z$  is positive. Thus deduce that this method is A-stable.

*Hint:* if  $z$  is on the boundary of the linear stability domain then  $\mathbf{y}_n = \exp(in\theta)$  for some real  $\theta$ .

10. The (stiff) differential equation

$$y'(t) = -10^4(y - t^{-1}) - t^{-2}, \quad t \geq 1, \quad y(1) = 1,$$

has the analytic solution  $y(t) = t^{-1}$ ,  $t \geq 1$ . Let it be solved numerically by Euler's method  $\mathbf{y}_{n+1} = \mathbf{y}_n + h_n\mathbf{f}(t_n, \mathbf{y}_n)$  and the backward Euler method  $\mathbf{y}_{n+1} = \mathbf{y}_n + h_n\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})$ , where  $h_n = t_{n+1} - t_n$  is allowed to depend on  $n$  and to be different in the two cases. Suppose that, for any  $t_n \geq 1$ , we have  $|y_n - y(t_n)| \leq 10^{-6}$ , and that we require  $|y_{n+1} - y(t_{n+1})| \leq 10^{-6}$ . Show that Euler's method can fail if  $h_n = 2 \times 10^{-4}$ , but that the backward Euler method always succeeds if  $h_n \leq 10^{-2}t_n t_{n+1}^2$ .

*Hint:* find relations between  $y_{n+1} - y(t_{n+1})$  and  $y_n - y(t_n)$  for general  $y_n$  and  $t_n$ .

11. This question concerns the predictor-corrector pair

$$\mathbf{y}_{n+3}^P = -\frac{1}{2}\mathbf{y}_n + 3\mathbf{y}_{n+1} - \frac{3}{2}\mathbf{y}_{n+2} + 3h\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}),$$

$$\mathbf{y}_{n+3}^C = \frac{1}{11}[2\mathbf{y}_n - 9\mathbf{y}_{n+1} + 18\mathbf{y}_{n+2} + 6h\mathbf{f}(t_{n+3}, \mathbf{y}_{n+3})].$$

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne's device has the value  $\frac{6}{17}|\mathbf{y}_{n+3}^P - \mathbf{y}_{n+3}^C|$ .

12. Let  $u(x)$ ,  $0 \leq x \leq 1$ , be a six-times differentiable function that satisfies the ODE  $u''(x) = f(x)$ ,  $0 \leq x \leq 1$ ,  $u(0)$  and  $u(1)$  being given. Further, we let  $x_m = mh = m/M$ ,  $m = 0, 1, \dots, M$ , for some positive integer  $M$ , and calculate the estimates  $u_m \approx u(x_m)$ ,  $m = 1, 2, \dots, M-1$ , by solving the difference equation

$$u_{m-1} - 2u_m + u_{m+1} = h^2 f(x_m) + \alpha h^2 [f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \quad m = 1, 2, \dots, M-1,$$

where  $u_0 = u(0)$ ,  $u_M = u(1)$ , and  $\alpha$  is a positive parameter. Show that there exists a choice of  $\alpha$  such that the local truncation error of the difference equation is  $\mathcal{O}(h^6)$ .