

B13b

Numerical Analysis: Example Sheet 2

Lent 2016

A * denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** necessarily harder than unstarred questions.

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1. Let $h = 1/M$, where $M \geq 1$ is an integer. Consider the differential equations

$$y' = -\frac{y}{1+t} \quad \text{and} \quad y' = \frac{2y}{1+t}, \quad 0 \leq t \leq 1, \quad (*)$$

with initial condition $y(0) = 1$ in both cases. In each case find the exact solution of the differential equation.

Define Euler's method, and apply it to calculate the estimates $\{\mathbf{y}_n\}_{n=1,2,\dots,M}$ of $\mathbf{y}(nh)$ for each of the equations letting $y_0 = y(0) = 1$. By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for y_n , $n = 1, 2, \dots, M$, which should be free from summations and products of n terms. Deduce the exact solutions of the equations from the limit $h \rightarrow 0$. Verify that the magnitude of the errors $y_n - y(nh)$, $n = 1, 2, \dots, M$, is at most $\mathcal{O}(h)$.

2. Assuming that \mathbf{f} satisfies the Lipschitz condition and the true solution possesses a bounded third derivative in $[0, t^*]$, apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]$$

converges and that $\|\mathbf{y}_n - \mathbf{y}(t_n)\| \leq ch^2$ for some $c > 0$ and all n such that $0 \leq nh \leq t^*$.

3. The s -step Adams–Bashforth method is of order s and has the form

$$\mathbf{y}_{n+s} = \mathbf{y}_{n+s-1} + h \sum_{j=0}^{s-1} \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j}).$$

Calculate the actual values of the coefficients in the case $s = 3$

4. By solving a three-term recurrence relation, calculate analytically the sequence of values $\{\mathbf{y}_n : n = 2, 3, 4, \dots\}$ that is generated by the *explicit midpoint rule*

$$\mathbf{y}_{n+2} = \mathbf{y}_n + 2h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}),$$

when it is applied to the ODE $y' = -y$, $t \geq 0$. Starting from the values $y_0 = 1$ and $y_1 = 1 - h$, show that the sequence diverges as $n \rightarrow \infty$ for *all* $h > 0$.

However if the order of the method is greater or equal to one, the root condition and suitable starting conditions imply convergence in a *finite* interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem.

Hint: in the last part, relate the roots of the recurrence relation to $\pm e^{\pm h} + \mathcal{O}(h^3)$.

5. Show that the multistep method

$$\sum_{j=0}^3 \rho_j \mathbf{y}_{n+j} = h \sum_{j=0}^2 \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j}), \quad \text{where } \rho_3 = 1 \text{ (as in lectures),}$$

is fourth order only if the conditions $\rho_0 + \rho_2 = 8$ and $\rho_1 = -9$ are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition

- *6. An s -stage explicit Runge–Kutta method of order s with constant step size $h > 0$ is applied to the differential equation $y' = \lambda y$, $t \geq 0$. Prove the identity

$$y_n = \left[\sum_{\ell=0}^s \frac{1}{\ell!} (h\lambda)^\ell \right]^n y_0, \quad n = 0, 1, 2, \dots,$$

by showing by induction that $k_i = \lambda y_n p_{i-1}(\lambda h)$ where the p_{i-1} are polynomials of degree $i - 1$, deducing that $y_{n+1} = p_s(\lambda h)y_n$, and using the order condition to determine p_s ; or otherwise.

7. The following four-stage Runge–Kutta method has order four,

$$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n)$$

$$\mathbf{k}_2 = \mathbf{f}(t_n + \frac{1}{3}h, \mathbf{y}_n + \frac{1}{3}h\mathbf{k}_1)$$

$$\mathbf{k}_3 = \mathbf{f}(t_n + \frac{2}{3}h, \mathbf{y}_n - \frac{1}{3}h\mathbf{k}_1 + h\mathbf{k}_2)$$

$$\mathbf{k}_4 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_1 - h\mathbf{k}_2 + h\mathbf{k}_3)$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h(\frac{1}{8}\mathbf{k}_1 + \frac{3}{8}\mathbf{k}_2 + \frac{3}{8}\mathbf{k}_3 + \frac{1}{8}\mathbf{k}_4).$$

By considering the equation $y' = y$, show that the order is at most four. Then, for scalar functions, prove that the order is at least four in the easy case when f is independent of y , and that the order is at least three in the relatively easy case when f is independent of t .

Comment: do not derive all of the (gory) details when $f(t, y)$ depends on both t and \mathbf{y} .

8. Find $\mathcal{D} \cap \mathbb{R}$, the intersection of the linear stability domain \mathcal{D} with the real axis, for the following methods:

$$(1) \mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_n, \mathbf{y}_n) \quad (2) \mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]$$

$$(3) \mathbf{y}_{n+2} = \mathbf{y}_n + 2h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) \quad (4) \mathbf{y}_{n+2} = \mathbf{y}_{n+1} + \frac{1}{2}h[3\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - \mathbf{f}(t_n, \mathbf{y}_n)]$$

$$(5) \text{ The RK method } \mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n), \quad \mathbf{k}_2 = \mathbf{f}(t_n + h, \mathbf{y}_n + h\mathbf{k}_1), \quad \mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h(\mathbf{k}_1 + \mathbf{k}_2).$$

Hint: note that to solve $a < X < b$, instead of manipulating the inequalities it can be easier to solve $X = a$ and $X = b$, and then decide which region is required by considering one interior point (as for conformal maps).

9. Show that, if z is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method

$$\mathbf{y}_{n+2} - \frac{4}{3}\mathbf{y}_{n+1} + \frac{1}{3}\mathbf{y}_n = \frac{2}{3}h\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})$$

then the real part of z is positive. Thus deduce that this method is A-stable.

Hint: if z is on the boundary of the linear stability domain then $y_n = \exp(in\theta)$ for some real θ .

10. The (stiff) differential equation

$$y'(t) = -10^4(y - t^{-1}) - t^{-2}, \quad t \geq 1, \quad y(1) = 1,$$

has the analytic solution $y(t) = t^{-1}$, $t \geq 1$. Let it be solved numerically by Euler's method $y_{n+1} = y_n + h_n f(t_n, y_n)$ and the backward Euler method $y_{n+1} = y_n + h_n f(t_{n+1}, y_{n+1})$, where $h_n = t_{n+1} - t_n$ is allowed to depend on n and to be different in the two cases. Suppose that, for any $t_n \geq 1$, we have $|y_n - y(t_n)| \leq 10^{-6}$, and that we require $|y_{n+1} - y(t_{n+1})| \leq 10^{-6}$. Show that Euler's method can fail if $h_n = 2 \times 10^{-4}$, but that the backward Euler method always succeeds if $h_n \leq 10^{-2}t_n t_{n+1}^2$.

Hint: find relations between $y_{n+1} - y(t_{n+1})$ and $y_n - y(t_n)$ for general y_n and t_n .

11. This question concerns the predictor-corrector pair

$$\mathbf{y}_{n+3}^P = -\frac{1}{2}\mathbf{y}_n + 3\mathbf{y}_{n+1} - \frac{3}{2}\mathbf{y}_{n+2} + 3h\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}),$$

$$\mathbf{y}_{n+3}^C = \frac{1}{11}[2\mathbf{y}_n - 9\mathbf{y}_{n+1} + 18\mathbf{y}_{n+2} + 6h\mathbf{f}(t_{n+3}, \mathbf{y}_{n+3})].$$

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne's device has the value $\frac{6}{17}|\mathbf{y}_{n+3}^P - \mathbf{y}_{n+3}^C|$.

12. Let $u(x)$, $0 \leq x \leq 1$, be a six-times differentiable function that satisfies the ODE $u''(x) = f(x)$, $0 \leq x \leq 1$, $u(0)$ and $u(1)$ being given. Further, we let $x_m = mh = m/M$, $m = 0, 1, \dots, M$, for some positive integer M , and calculate the estimates $u_m \approx u(x_m)$, $m = 1, 2, \dots, M-1$, by solving the difference equation

$$u_{m-1} - 2u_m + u_{m+1} = h^2 f(x_m) + \alpha h^2 [f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \quad m = 1, 2, \dots, M-1,$$

where $u_0 = u(0)$, $u_M = u(1)$, and α is a positive parameter. Show that there exists a choice of α such that the local truncation error of the difference equation is $\mathcal{O}(h^6)$.