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## Mathematical Tripos Part IB: Lent Term 2024

# Numerical Analysis – Examples' Sheet 2

**1.** Given $h > 0$, let Euler's method be applied to calculate the estimates $\{y_n\}$ of $y(nh)$ for each of the differential equations

$$y' = -\frac{y}{1+t} \quad \text{and} \quad y' = \frac{2y}{1+t}, \qquad 0 \le t \le 1,$$

starting with $y_0 = y(0) = 1$ in both cases. By using induction and by cancelling as many terms as possible in the resultant products, deduce simple explicit expressions for $y_n$, which should be free from summations and products of $n$ terms. Hence deduce the exact solutions of the equations from the limit $h \to 0$ and $nh \to t$. Verify that, for $nh \in [0,1]$, the magnitude of the errors $y_n - y(nh)$ is at most $\mathcal{O}(h)$.

**2.** Assuming that $\boldsymbol{f}$ satisfies the Lipschitz condition and possesses a bounded third derivative in $[0,T]$, apply the method of analysis of the Euler method, given in the lectures, to prove that the trapezoidal rule

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \tfrac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$$

converges and that $\|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| \le ch^2$ for some $c > 0$ and all $n$ such that $0 \le nh \le T$.

**3.** The $s$-step Adams–Bashforth method is of order $s$ and has the form

$$\boldsymbol{y}_{n+s} = \boldsymbol{y}_{n+s-1} + h \sum_{m=0}^{s-1} \boldsymbol{b}_m \boldsymbol{f}(t_{n+m}, \boldsymbol{y}_{n+m}).$$

Calculate the actual values of the coefficients in the case $s = 3$.

**4.** By solving a three-term recurrence relation, calculate analytically the sequence of values $\{\boldsymbol{y}_n\}_{n\ge 2}$ that is generated by the *explicit midpoint rule*

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}),$$

when it is applied to the ODE $y' = -y$, $t \ge 0$. Starting from the values $y_0 = 1$ and $y_1 = 1 - h$, show that the sequence diverges as $n \to \infty$ for *any* $h > 0$. Recall, however, that order $\ge 1$, the root condition and suitable starting conditions imply convergence in a *finite* interval. Prove that the above implementation of the explicit midpoint rule is consistent with this theorem.

[*Hint*. In the last part, relate the roots of the recurrence relation to $\pm e^{\mp h} + \mathcal{O}(h^3)$.]

**5.** Show that the multistep method (with $a_3 = 1$)

$$\sum_{m=0}^{3} a_m \boldsymbol{y}_{n+m} = h \sum_{m=0}^{2} b_m \boldsymbol{f}(t_{n+m}, \boldsymbol{y}_{n+m})$$

is fourth order only if the conditions $a_0 + a_2 = 8$ and $a_1 = -9$ are satisfied. Hence deduce that this method cannot be both fourth order and satisfy the root condition. (Compare this conclusion with Theorem 8.9.)

**6.** An $s$-stage explicit Runge–Kutta method with constant step size $h > 0$ is applied to the differential equation $y' = \lambda y$, $t \ge 0$. Prove that, with some polynomial $p_s$ of degree $s$, we have

$$y_{n+1} = p_s(\lambda h) y_n, \qquad p_s \in \mathcal{P}_s\,.$$

Hence derive that no explicit Runge-Kutta method can be A-stable (or even $\text{A}_0$-stable). Further, prove that if an $s$-stage RK method is of order $s$, then

$$p_s(\lambda h) = \sum_{m=0}^{s} \frac{1}{m!} (\lambda h)^m\,.$$

**7.** The following four-stage Runge–Kutta method has order four,

$$\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{f}(t_n, \boldsymbol{y}_n) \\
\boldsymbol{k}_2 &= \boldsymbol{f}(t_n + \tfrac{1}{3}h, \boldsymbol{y}_n + \tfrac{1}{3}h\boldsymbol{k}_1) \\
\boldsymbol{k}_3 &= \boldsymbol{f}(t_n + \tfrac{2}{3}h, \boldsymbol{y}_n - \tfrac{1}{3}h\boldsymbol{k}_1 + h\boldsymbol{k}_2) \\
\boldsymbol{k}_4 &= \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1 - h\boldsymbol{k}_2 + h\boldsymbol{k}_3) \\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h\left(\tfrac{1}{8}\boldsymbol{k}_1 + \tfrac{3}{8}\boldsymbol{k}_2 + \tfrac{3}{8}\boldsymbol{k}_3 + \tfrac{1}{8}\boldsymbol{k}_4\right).
\end{aligned}$$

By considering the equation $y' = y$, show that the order is at most four. Then, for scalar $f$, prove that the order is at least four in the case when $f$ is independent of $y$. (In the last case, you may relate the scheme to a quadrature formula.)

**8.** Find $\mathcal{D} \cap \mathbb{R}$, the intersection of the linear stability domain $\mathcal{D}$ with the real axis, for the following methods:

(1) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$

(2) $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$

(3) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})$

(4) $\boldsymbol{y}_{n+2} = \boldsymbol{y}_{n+1} + \frac{1}{2}h[3\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) - \boldsymbol{f}(t_n, \boldsymbol{y}_n)]$

(5) The RK method: $\boldsymbol{k}_1 = \boldsymbol{f}(t_n, \boldsymbol{y}_n)$, $\quad \boldsymbol{k}_2 = \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h\boldsymbol{k}_1)$, $\quad \boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h(\boldsymbol{k}_1 + \boldsymbol{k}_2)$.

**9.** Show that, if $z$ is a nonzero complex number that is on the boundary of the linear stability domain of the two-step BDF method

$$\boldsymbol{y}_{n+2} - \tfrac{4}{3}\boldsymbol{y}_{n+1} + \tfrac{1}{3}\boldsymbol{y}_n = \tfrac{2}{3}hf(t_{n+2}, \boldsymbol{y}_{n+2})\,,$$

then the real part of $z$ is positive. Thus deduce that this method is A-stable.

**10.** The (stiff) differential equation

$$y'(t) = -10^4(y - t^{-1}) - t^{-2}, \qquad t \geq 1, \qquad y(1) = 1,$$

has the analytic solution $y(t) = t^{-1}$, $t \geq 1$. Let it be solved numerically by Euler's method $y_{n+1} = y_n + h_n f(t_n, y_n)$ and the backward Euler method $y_{n+1} = y_n + h_n f(t_{n+1}, y_{n+1})$, where $h_n = t_{n+1} - t_n$ is allowed to depend on $n$ and to be different in the two cases. Suppose that, for any $t_n \geq 1$, we have $|y_n - y(t_n)| \leq 10^{-6}$, and that we require $|y_{n+1} - y(t_{n+1})| \leq 10^{-6}$. Show that Euler's method can fail if $h_n = 2 \times 10^{-4}$, but that the backward Euler method always succeeds if $h_n \leq 10^{-2} t_n t_{n+1}^2$.

*Hint*: Find relations between $y_{n+1} - y(t_{n+1})$ and $y_n - y(t_n)$ for general $y_n$ and $t_n$.

**11.** This question concerns the predictor-corrector pair

$$\begin{aligned}
\boldsymbol{y}^{\mathrm{P}}_{n+3} &= -\tfrac{1}{2}\boldsymbol{y}_n + 3\boldsymbol{y}_{n+1} - \tfrac{3}{2}\boldsymbol{y}_{n+2} + 3h\boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2}), \\
\boldsymbol{y}^{\mathrm{C}}_{n+3} &= \tfrac{1}{11}[2\boldsymbol{y}_n - 9\boldsymbol{y}_{n+1} + 18\boldsymbol{y}_{n+2} + 6h\boldsymbol{f}(t_{n+3}, \boldsymbol{y}_{n+3})].
\end{aligned}$$

Show that both methods are third order, and that the estimate of the error of the corrector formula by Milne's device has the value $\frac{6}{17}|\boldsymbol{y}^{\mathrm{P}}_{n+3} - \boldsymbol{y}^{\mathrm{C}}_{n+3}|$.

**12.** Let $u(x)$, $0 \leq x \leq 1$, be a six-times differentiable function that satisfies the ODE $u''(x) = f(x)$, $0 \leq x \leq 1$, $u(0)$ and $u(1)$ being given. Further, we let $x_m = mh = m/M$, $m = 0, 1, \ldots, M$, for some positive integer $M$, and calculate the estimates $u_m \approx u(x_m)$, $m = 1, 2, \ldots, M-1$, by solving the difference equation

$$u_{m-1} - 2u_m + u_{m+1} = h^2 f(x_m) + \alpha h^2[f(x_{m-1}) - 2f(x_m) + f(x_{m+1})], \qquad m = 1, 2, \ldots, M-1,$$

where $u_0 = u(0)$, $u_M = u(1)$, and $\alpha$ is a positive parameter. Show that there exists a choice of $\alpha$ such that the local truncation error of the difference equation is $\mathcal{O}(h^6)$.