1. Calculate all LU factorizations of the matrix

\[
A = \begin{bmatrix}
10 & 6 & -2 & 1 \\
10 & 10 & -5 & 0 \\
-2 & 2 & -2 & 1 \\
1 & 3 & -2 & 3 \\
\end{bmatrix},
\]

where all diagonal elements of \( L \) are one. By using one of these factorizations, find all solutions of the equation \( Ax = b \) where \( b^\top = [-2, 0, 2, 1] \).

2. By using column pivoting if necessary to exchange rows of \( A \), an LU factorization of a real \( n \times n \) matrix \( A \) is calculated, where \( L \) has ones on its diagonal, and where the moduli of the off-diagonal elements of \( L \) do not exceed one. Let \( \alpha \) be the largest of the moduli of the elements of \( A \). Prove by induction on \( i \) that elements of \( U \) satisfy the condition

\[ |u_{ij}| \leq 2^{i-1}\alpha. \]

Then construct 2 \( \times \) 2 and 3 \( \times \) 3 nonzero matrices \( A \) that yield \( |u_{22}| = 2\alpha \) and \( |u_{33}| = 4\alpha \) respectively.

3. Let \( A \) be a real \( n \times n \) matrix that has the factorization \( A = LU \), where \( L \) is lower triangular with ones on its diagonal and \( U \) is upper triangular. Prove that, for every integer \( k \in \{1, 2, \ldots, n\} \), the first \( k \) rows of \( U \) span the same space as the first \( k \) rows of \( A \). Prove also that the first \( k \) columns of \( A \) are in the \( k \)-dimensional subspace that is spanned by the first \( k \) columns of \( L \).

Hence deduce that no LU factorization of the given form exists if we have rank \( H_k < \text{rank} \ B_k \), where \( H_k \) is the leading \( k \times k \) submatrix of \( A \) and where \( B_k \) is the \( n \times k \) matrix whose columns are the first \( k \) columns of \( A \).

Further, deduce that if \( A \) is non-singular and the LU factorisation exists, then all leading zeros to the left of the diagonal in the rows of \( A \) are inherited by \( L \), and all leading zeros above the diagonal in the columns of \( A \) are inherited by \( U \).

4. Calculate the Cholesky factorization of the matrix

\[
\begin{bmatrix}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 1 \\
1 & 4 & 1 \\
1 & 5 & 1 \\
1 & \lambda \\
\end{bmatrix}
\]

Deduce from the factorization the value of \( \lambda \) that makes the matrix singular. Also find this value of \( \lambda \) by seeking the vector in the null-space of the matrix whose first component is one.
5. Let $A$ be an $n \times n$ nonsingular band matrix that satisfies the condition $a_{ij} = 0$ if $|i - j| > r$, where $r$ is small, and let Gaussian elimination with column pivoting be used to solve $Ax = b$. Identify all the coefficients of the intermediate equations that can become nonzero. Hence deduce that the total number of additions and multiplications of the complete calculation can be bounded by a constant multiple of $nr^2$.

6. Let $a_1$, $a_2$ and $a_3$ denote the columns of the matrix $A = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Apply the Gram–Schmidt procedure to $A$, which generates orthonormal vectors $q_1$, $q_2$ and $q_3$. Note that this calculation provides real numbers $r_{jk}$ such that $a_k = \sum_{j=1}^{k} r_{jk} q_j$, $k = 1, 2, 3$. Hence express $A$ as the product $A = QR$, where $Q$ and $R$ are orthogonal and upper-triangular matrices respectively.

7. Calculate the QR factorization of the matrix of question 6 by using three Givens rotations. Explain why the initial rotation can be any one of the three types $\Omega^{(1,2)}$, $\Omega^{(1,3)}$ and $\Omega^{(2,3)}$. Prove that the final factorization is independent of this initial choice in exact arithmetic, provided that we satisfy the condition that in each row of $R$ the leading nonzero element is positive.

8. Let $A$ be an $n \times n$ matrix, and for $i = 1, 2, \ldots, n$ let $k(i)$ be the number of zero elements in the $i$-th row of $A$ that come before all nonzero elements in this row and before the diagonal element $a_{ii}$. Show that the QR factorization of $A$ can be calculated by using at most $\frac{1}{2}n(n-1) - \sum k(i)$ Givens rotations. Hence show that, if $A$ is an upper triangular matrix except that there are nonzero elements in its first column, i.e. $a_{ij} = 0$ when $2 \leq j < i \leq n$, then its QR factorization can be calculated by using only $2n - 3$ Givens rotations.

*Hint*: you should find the order of the first $(n-2)$ rotations that brings your matrix to the form considered above.

9. Calculate the QR factorization of the matrix of question 6 by using two Householder reflections. Show that, if this technique is used to generate the QR factorization of a general $n \times n$ matrix $A$, then the computation can be organised so that the total number of additions and multiplications is bounded above by a constant multiple of $n^3$.

10. Let

$$A = \begin{bmatrix} 3 & 4 & 7 & -2 \\ 5 & 4 & 9 & 3 \\ 1 & -1 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ 29 \\ 16 \\ 10 \end{bmatrix}.$$

Calculate the QR factorization of $A$ by using Householder reflections. In this case $A$ is singular and you should choose $Q$ so that the last row of $R$ is zero. Hence identify all the least squares solutions of the inconsistent system $Ax = b$, where we require $x$ to minimize $||Ax - b||_2$. Verify that all the solutions give the same vector of residuals $Ax - b$, and that this vector is orthogonal to the columns of $A$. There is no need to calculate the elements of $Q$ explicitly.