

## Mathematical Tripos Part IB: Lent 2018

### Numerical Analysis – Exercise Sheet 3<sup>1</sup>

1. Calculate *all* LU factorizations of the matrix

$$A = \begin{bmatrix} 10 & 6 & -2 & 1 \\ 10 & 10 & -5 & 0 \\ -2 & 2 & -2 & 1 \\ 1 & 3 & -2 & 3 \end{bmatrix},$$

where all diagonal elements of  $L$  are one. By using one of these factorizations, find *all* solutions of the equation  $A\mathbf{x} = \mathbf{b}$  where  $\mathbf{b}^\top = [-2, 0, 2, 1]$ .

2. By using column pivoting if necessary to exchange rows of  $A$ , an LU factorization of a real  $n \times n$  matrix  $A$  is calculated, where  $L$  has ones on its diagonal, and where the moduli of the off-diagonal elements of  $L$  do not exceed one. Let  $\alpha$  be the largest of the moduli of the elements of  $A$ . Prove by induction on  $i$  that elements of  $U$  satisfy the condition  $|u_{ij}| \leq 2^{i-1}\alpha$ . Then construct  $2 \times 2$  and  $3 \times 3$  nonzero matrices  $A$  that yield  $|u_{22}| = 2\alpha$  and  $|u_{33}| = 4\alpha$  respectively.

3. Let  $A$  be a real  $n \times n$  matrix that has the factorization  $A = LU$ , where  $L$  is lower triangular with ones on its diagonal and  $U$  is upper triangular. Prove that, for every integer  $k \in \{1, 2, \dots, n\}$ , the first  $k$  rows of  $U$  span the same space as the first  $k$  rows of  $A$ . Prove also that the first  $k$  columns of  $A$  are in the  $k$ -dimensional subspace that is spanned by the first  $k$  columns of  $L$ . Hence deduce that no LU factorization of the given form exists if we have  $\text{rank } H_k < \text{rank } B_k$ , where  $H_k$  is the leading  $k \times k$  submatrix of  $A$  and where  $B_k$  is the  $n \times k$  matrix whose columns are the first  $k$  columns of  $A$ .

4. Calculate the Cholesky factorization of the matrix

$$\begin{bmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{bmatrix}.$$

Deduce from the factorization the value of  $\lambda$  that makes the matrix singular. Also find this value of  $\lambda$  by seeking the vector in the null-space of the matrix whose first component is one.

5. Let  $A$  be an  $n \times n$  nonsingular band matrix that satisfies the condition  $a_{ij} = 0$  if  $|i-j| > r$ , where  $r$  is small, and let Gaussian elimination *with column pivoting* be used to solve  $Ax = b$ . Identify all the coefficients of the intermediate equations that can become nonzero. Hence deduce that the total number of additions and multiplications of the complete calculation can be bounded by a constant multiple of  $nr^2$ .

<sup>1</sup>Corrections and suggestions should be emailed to [h.fawzi@damtp.cam.ac.uk](mailto:h.fawzi@damtp.cam.ac.uk).

6. Let  $a_1$ ,  $a_2$  and  $a_3$  denote the columns of the matrix

$$A = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Apply the Gram–Schmidt procedure to  $A$ , which generates orthonormal vectors  $\mathbf{q}_1$ ,  $\mathbf{q}_2$  and  $\mathbf{q}_3$ . Note that this calculation provides real numbers  $r_{jk}$  such that  $\mathbf{a}_k = \sum_{j=1}^k r_{jk} \mathbf{q}_j$ ,  $k = 1, 2, 3$ . Hence express  $A$  as the product  $A = QR$ , where  $Q$  and  $R$  are orthogonal and upper-triangular matrices respectively.

7. Calculate the QR factorization of the matrix of question 6 by using three Givens rotations. Explain why the initial rotation can be any one of the three types  $\Omega^{(1,2)}$ ,  $\Omega^{(1,3)}$  and  $\Omega^{(2,3)}$ . Prove that the final factorization is independent of this initial choice in exact arithmetic, provided that we satisfy the condition that in each row of  $R$  the leading nonzero element is positive.

8. Let  $A$  be an  $n \times n$  matrix, and for  $i = 1, 2, \dots, n$  let  $k(i)$  be the number of zero elements in the  $i$ -th row of  $A$  that come before all nonzero elements in this row and before the diagonal element  $a_{ii}$ . Show that the QR factorization of  $A$  can be calculated by using at most  $\frac{1}{2}n(n-1) - \sum k(i)$  Givens rotations. Hence show that, if  $A$  is an upper triangular matrix except that there are nonzero elements in its first column, i.e.  $a_{ij} = 0$  when  $2 \leq j < i \leq n$ , then its QR factorization can be calculated by using only  $2n - 3$  Givens rotations. [*Hint:* You should find the order of the first  $(n - 2)$  rotations that brings your matrix to the form considered above.]

9. Calculate the QR factorization of the matrix of question 6 by using two Householder reflections. Show that, if this technique is used to generate the QR factorization of a general  $n \times n$  matrix  $A$ , then the computation can be organised so that the total number of additions and multiplications is bounded above by a constant multiple of  $n^3$ .

10. Let

$$A = \begin{bmatrix} 3 & 4 & 7 & -2 \\ 5 & 4 & 9 & 3 \\ 1 & -1 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ 29 \\ 16 \\ 10 \end{bmatrix}.$$

Calculate the QR factorization of  $A$  by using Householder reflections. In this case  $A$  is singular and you should choose  $Q$  so that the last row of  $R$  is zero. Hence identify all the least squares solutions of the inconsistent system  $A\mathbf{x} = \mathbf{b}$ , where we require  $\mathbf{x}$  to minimize  $\|A\mathbf{x} - \mathbf{b}\|_2$ . Verify that all the solutions give the same vector of residuals  $A\mathbf{x} - \mathbf{b}$ , and that this vector is orthogonal to the columns of  $A$ . There is no need to calculate the elements of  $Q$  explicitly.