

**Example Sheet 2: Shocks and Elastic Waves<sup>‡</sup>**

1. *An expansion fan.* A piston confines an ideal, compressible fluid to the right-hand half,  $x > 0$ , of an infinite tube. The piston and the fluid are initially at rest for times  $t < 0$ . At time  $t = 0$  the piston starts moving with constant speed  $V$  away from the fluid. Show that there is just one region in the  $(x, t)$  plane,  $R_2$  say, throughout which the local sound speed  $c$  takes a constant value  $c_2$  different from its value  $c_0$  in the undisturbed region,  $R_0$  say. Show by *reductio ad absurdum*, or otherwise, that all the  $C_+$  characteristics lying outside both  $R_2$  and  $R_0$  must pass through the origin, and deduce that the form of the disturbance at time  $t > 0$  is

$$u + c = \left\{ \begin{array}{ll} c_2 - V, & -Vt \leq x \leq (c_2 - V)t \\ xt^{-1}, & (c_2 - V)t \leq x \leq c_0t \\ c_0, & x \geq c_0t \end{array} \right\}. \quad (7)$$

Sketch a graph of this *expansion fan* as a function of  $x$ . Would the individual graphs of  $u$  and  $c$  look the same as the graph for  $(u + c)$ : (a)\* for a fluid with  $p \propto (e^\rho - 1)$ , and (b) for a perfect gas with  $p \propto \rho^\gamma$ ?

For the case of a perfect gas explain why the solution (7) is physically meaningless when

$$V > 2(\gamma - 1)^{-1}c_0.$$

Deduce that a gas expanding one-dimensionally into a vacuum does so with the *escape speed*

$$2(\gamma - 1)^{-1}c_0.$$

2. *Two expansion fans* (Old Tripos: 75425). A perfect gas, with constant specific heats in the ratio  $\gamma$ , is initially at rest with uniform sound speed  $c_0$ . It is confined by two pistons to the region  $0 < x < 2\ell$  of a long cylindrical tube. At time  $t = 0$ , both pistons are set into impulsive motion away from the gas with constant velocities  $u = -V < 0$  and  $u = U > 0$ .

(i) For  $0 \leq t \leq \ell/c_0$  show that in the part  $x \leq \ell$  of the tube (which cannot have been reached by any signal from the piston initially at  $x = 2\ell$ ), every  $C_+$  characteristic is a straight line.<sup>†</sup> Show that the fluid velocity  $u$  takes the value

$$u = \frac{2}{\gamma + 1} \left( \frac{x}{t} - c_0 \right) \quad \text{for} \quad \left( c_0 - \frac{\gamma + 1}{2}V \right)t < x < c_0t.$$

Give the corresponding value of  $c$ . Find the shape of the  $C_-$  characteristics when  $u$  and  $c$  take these values.

(ii) Deduce that, when  $t > \ell/c_0$ , the equation

$$u = \frac{2}{\gamma + 1} \left( \frac{x}{t} - c_0 \right)$$

<sup>‡</sup> Corrections and suggestions should be emailed to S.J.Cowley@damtp.cam.ac.uk.

<sup>†</sup> Results derived in question 1 should be quoted.

is satisfied only in the smaller interval

$$\left(c_0 - \frac{\gamma + 1}{2}V\right)t < x < \frac{\ell}{\gamma - 1} \left( (\gamma + 1) \left( \frac{c_0 t}{\ell} \right)^{(3-\gamma)/(\gamma+1)} - 2 \left( \frac{c_0 t}{\ell} \right) \right).$$

(iii) For a case with  $V/c_0$  about  $\frac{1}{2}$  and  $U/c_0$  about  $\frac{1}{4}$ , give a rough sketch indicating **four** areas of the  $(x, t)$  plane throughout each of which  $u$  takes a different constant value, to be specified.

3. *A piston-generated shock.* A piston moves with constant positive velocity  $u_1$  into a perfect gas of specific heat ratio  $\gamma > 1$ , generating a shock wave which moves ahead of the piston. Show that a possible solution of all the relevant equations is one in which the gas is at rest beyond the shock, at pressure  $p_0$ , and is moving with constant velocity  $u_1$  in the region between the piston and the shock, throughout which region the density and pressure also take constant values  $\rho_1, p_1$  which are determined by

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta}, \quad \frac{1}{\beta^2} + \frac{\gamma + 1}{2\gamma\beta} = \frac{c_0^2}{\gamma^2 u_1^2},$$

where  $\beta$  is the shock strength defined as  $(p_1 - p_0)/p_0 > 0$ , and  $\rho_0$  and  $c_0$  are the density and sound speed of the undisturbed gas. The internal energy per unit mass of a perfect gas of density  $\rho$  at pressure  $p$  equals  $p/(\gamma - 1)\rho$ .

*Hint.* In order to demonstrate that the propagation speed  $C$  of the shock does indeed exceed the given velocity  $u_1$ , it may be helpful to show that

$$C = c_0 \left( 1 + \frac{\gamma + 1}{2\gamma} \beta \right)^{1/2}.$$

4.\* *A method to generate shock waves in a 'shock tube'* (Old Tripos: 85427). An infinitely long uniform tube contains two perfect gases separated by a membrane at  $x = 0$ . The gas in  $x > 0$  has pressure  $p_1$ , density  $\rho_1$  and specific heat ratio  $\gamma_1$ ; the corresponding values for the gas in  $x < 0$  are  $p_2, \rho_2, \gamma_2$  where  $p_2 > p_1$ . At  $t = 0$  the membrane is burst. Assuming that the interface between the two gases remains plane and moves with constant speed  $V$ , use the one-dimensional equations of motion to show that there are three regions in the tube in which the pressure is uniform,

$$\begin{aligned} p &= p_2 \quad \text{for } x < - \left( \frac{\gamma_2 p_2}{\rho_2} \right)^{1/2} t, \\ p &= p_1 \quad \text{for } x > Ut, \\ p &= p_m \quad \text{for } - \left[ \left( \frac{\gamma_2 p_2}{\rho_2} \right)^{1/2} - \frac{\gamma_2 + 1}{2} V \right] t < x < Ut, \end{aligned}$$

where  $p_m$  is as yet unknown, and the shock velocity,  $U$ , is a constant to be found in terms of  $p_m, p_1, \rho_1, \gamma_1$ .

Show that  $V$  is related to  $p_m$  by the following two equations:

$$\begin{aligned} V &= (p_m - p_1) \left( \frac{1}{2} \rho_1 [(\gamma_1 + 1)p_m + (\gamma_1 - 1)p_1] \right)^{-1/2}, \\ V &= \frac{2}{\gamma_2 - 1} \left( \frac{\gamma_2 p_2}{\rho_2} \right)^{1/2} \left[ 1 - \left( \frac{p_m}{p_2} \right)^{(\gamma_2 - 1)/2\gamma_2} \right], \end{aligned}$$

and hence show that there is a unique solution for  $p_m$  and  $V$ .

5. *Generalisation of Hooke's law.* For an uniaxial tension,

$$\sigma_{11} = \sigma, \quad \sigma_{ij} = 0 \quad \text{for } i \neq 1, j \neq 1,$$

show that  $E\mathbf{u} = \sigma(x, -\nu y, -\nu z)$ , where the Young's modulus  $E$ , and Poisson's ratio  $\nu$ , should be expressed in terms of the bulk modulus and shear modulus. For what range of values of  $E$  and  $\nu$  is the elastic energy density positive definite? [In practice  $\nu > 0$ , so that an axial strain induces a radial contraction.]

Typical values of  $E$  (in units of  $10^{10} \text{ Nm}^{-2}$ ) and  $\nu$  for materials at  $20^\circ$  are:

	$E$	$\nu$		$E$	$\nu$
Aluminium	7.0	0.35	Soft iron	12.1	0.29
Copper	13.0	0.34	Mild steel	21.2	0.29

A cylindrical steel wire of unstrained radius 1 mm and length 1 m has a weight of 1 kg suspended from it. Find the increase in length, and the decrease in diameter, of the wire. [Very high forces generate very small strains — hence in engineering practice linear theory is very good. Beyond modest strains, e.g.  $10^{-2}$ , metals turn plastic, and the engineers normally redesign their structures.]

6. *A hanging rod.* [Optional, since the question has little to do with waves.] A circular homogeneous cylinder of length  $l$ , radius  $a$  and density  $\rho$  hangs under its own weight. Verify that, under suitable boundary conditions (to be specified), an appropriate displacement field is

$$\mathbf{u} = \frac{\rho g}{E} (\nu x(z-l), \nu y(z-l), lz - \frac{1}{2}z^2 - \frac{1}{2}\nu(x^2 + y^2)) ,$$

where  $E$  and  $\nu$  are the Young's modulus and Poisson's ratio for the solid. Sketch the deformed shape, and estimate the size of the effect for an iron beam. How long must the beam be for yield to occur?

7. *Energy fluxes.* Find the energy flux vectors both for a plane harmonic P-wave, and for a plane harmonic S-wave, travelling on its own in homogeneous material. Find also the energy flux for the case where both waves travel together in the same direction. Can the separate energy flux vectors be added to give the flux in the second case?

8. *Reflection of a SV-wave.* A solid with elastic wavespeeds  $c_P, c_S$  occupies the region  $y < 0$  and is bonded to a rigid barrier  $y = 0$ . A SV-wave with elastodynamic potential

$$\psi = Ae^{i(\omega t - kx \sin \theta - ky \cos \theta)} \mathbf{e}_z ,$$

is incident on the barrier. If  $\sin \theta < q = c_S/c_P$ , obtain the reflected waves. If  $\sin \theta > q$  show that a solution may be found consisting of a reflected SV-wave together with a compressional interface wave near  $y = 0$ .