1. At how many points does the function
   \[ \phi(x_1, x_2, x_3) = \frac{1}{2}(x_1^4 + x_2^4 + x_3^4) - x_2 x_3 - x_3 x_1 - x_1 x_2 \]
   take its minimum value? Show that this least value is \(-3\). By considering the eigenvalues of the
   Hessian, show also that \(\phi\) has one saddle point, at which the surface of vanishing \(\phi\) is tangent to a
   double cone of semi-angle \(\arctan \sqrt{2}\).

2. Show that
   (i) \(x^2/y\) is convex on the upper half plane \((x, y) : y > 0\).
   (ii) Show that if \(f(x)\) is convex then the function \(y f(y^{-1}x)\) is convex on \((x, y) : y > 0\).

3. Find the Legendre transform of \(f(x) = e^x\), (giving its domain also). Find the Legendre transform of
   \(f(x) = a^{-1}x^2, a > 1\) defined on \(x > 0\), and hence deduce Young’s inequality
   \[ xy \leq \frac{x^a}{a} + \frac{y^b}{b}, \quad \frac{1}{a} + \frac{1}{b} = 1. \]

4. For an ideal gas, the internal energy \(U = U(S, V)\) as a function of entropy and volume is
   \[ U = U_0 + a n R T \left[ \left( \frac{V_0}{V} \right)^{\frac{1}{2}} e^{\frac{S - S_0}{k T}} - 1 \right] \]
   for some constants \(U_0, T_0, V_0, S_0, a, n, R\). Calculate the Helmholtz free energy \(F = F(T, V)\) defined
   by \(F(T, V) = \min_S(U(S, V) - TS)\).

5. The area \(A\) of a triangle with sides \(a, b, c\) is given by
   \[ A = \sqrt{s[(s - a)(s - b)(s - c)]}, \quad \text{where} \quad s = \frac{1}{2}(a + b + c). \]
   (i) Show that of all triangles of given perimeter \(2s\), the triangle of largest area is equilateral.
   (ii) Find (in terms of the perimeter) the largest possible area of a right-angled triangle of given
       perimeter.

6. Find the maximum volume of a rectangular parallelepiped inscribed inside the ellipsoid
   \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \]

7. Let \((\theta, \phi)\) be the standard angular coordinates on the unit sphere. A function \(\phi(\theta)\) defines a path on
   this sphere. Given that \(\theta\) increases by \(\delta \theta\) over a short segment of this path, show that the length of
   the path segment is \(\sqrt{1 + (\phi' \sin \theta)^2} \delta \theta\) to first order in \(\delta \theta\). Hence find a functional
   \(L[\phi]\) for the total length of a path between any two points on the unit sphere. Use your result to show that the paths
   of minimal length are segments of great circles.

8. A soap film is bounded by two circular wires at \(r = a, z = \pm b\) in cylindrical polar coordinates
   \((r, \theta, z)\). Given that the soap surface is cylindrically symmetric, show that the equation of the surface
   of minimal area is
   \[ r = c \cosh (z/c) \]
   where \(c\) satisfies the condition \(a/c = \cosh (b/c)\). Show graphically that this condition has no solution
   for \(c\) if \(b/a\) is larger that a certain critical ratio. What happens to the soap surface as \(b/a\) is increased
   from below this ratio to above it?

9. It has been suggested that Crossrail, the new rail connection under London, should include a friction-
   less tunnel in which fuel-less trains can run under gravity. The trains are released from rest at the
   point of departure (Stratford) and are then allowed to run freely until arriving at their destination
   (Acton) at the same level. Assuming that the acceleration \(g\) due to gravity is uniform, show that the
   minimum travel time is \(\sqrt{2 \pi \ell / g}\), where \(\ell\) is the horizontal distance between the departure and arrival
   points. Comment on the quality of the likely travel experience at the departure and arrival points.
10. In an optical medium filling the region $0 < y < h$, the speed of light is
\[ c(y) = \frac{c_0}{\sqrt{1 - ky}}, \quad 0 < k < 1/h. \]

Show that the paths of light rays in the medium are parabolic. Show also that if a ray enters the medium at $(-x_0, 0)$ and leaves it at $(x_0, 0)$ then
\[ (kx_0)^2 = 4ky_0(1 - ky_0), \]

where $y_0$ ($< h$) is the greatest value of $y$ attained on the ray path.

11. Determine all functions $u(x)$ that extremize the functional
\[ I[u] = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} u'^2 + (1 - \cos u) \right\} dx \quad (*) \]
subject to the boundary conditions
\[ \lim_{x \to -\infty} u(x) = 0, \quad \lim_{x \to \infty} u(x) = 2\pi. \]

For $u(x)$ satisfying these boundary conditions, show that
\[ I[u] = \frac{1}{2} \int_{-\infty}^{\infty} (u' - 2\sin(u/2))^2 dx + 8. \]

Deduce that $I[u] \leq 8$ for such functions, and write down a first-order differential equation that $u$ must satisfy in order to realise this lower bound. How is this differential equation related to the second-order Euler-Lagrange equation for the function $I[u]$ of $(*)$?

12. Dido’s problem. An area $A$ of a field is enclosed by a length $\ell$ of flexible fencing with its ends attached a distance $a$ apart on a straight wall, where $a < \ell < \frac{1}{2}\pi a$. Show that $A$ is maximised, for fixed $\ell$, by the arc of a circle, and derive an equation relating the radius of the circle to the location of its centre. Comment on the case of $\ell > \frac{1}{2}\pi a$.

13. A uniform cable of fixed length, suspended between the two points $(-a, b)$ and $(a, b)$ has potential energy
\[ V = \int_{-a}^{a} y \sqrt{1 + y'^2} dx. \]

Write down a functional for the total length and then use the Lagrange multiplier method to show that the curve $y(x)$ of minimum energy assumed by the cable is a catenary:
\[ y - y_0 = c \cosh \left( \frac{x - x_0}{c} \right), \]

where $x_0$, $y_0$ and $c$ are constants. Find an equation for $c$ and show that it has a unique positive solution.