Variational Principles: Example Sheet 1

David Stuart Easter 2009

Comments/corrections to dmas2@cam.ac.uk. Sheet for supervisors will be available. The starred questions are not required for supervision.

1. Prove that if $f \in C^1(\mathbb{R})$ has only one staionary point which is a local minimum, then it must be a global minimum. Give a counter-example to show this is false in $\mathbb{R}^2$.
2. Prove that a real symmetric matrix $A_{ij} > 0$ iff all its eigenvalues are positive.
3. * Prove, using the Bolzano-Weierstrass property, but without using diagonalization, that if a real symmetric matrix $A_{ij} > 0$ then $\sum_{ij} A_{ij} v^i v^j \geq c\|\mathbf{v}\|^2$ for some $c > 0$. (After analysis II).
4. Let $f \in C^2(\mathbb{R}^2)$ have a stationary point $\mathbf{x} = (x^1, x^2)$ and let $A_{ij} = \partial^2_{ij} f(\mathbf{x})$. Show that $A_{11} + A_{22} > 0$ and $A_{11}A_{22} - A_{12}^2 > 0$ implies $A_{ij} > 0$ so that $\mathbf{x}$ is a strict local minimum.
5. Given $f : \mathbb{R}^n \to \mathbb{R}$ define its epigraph to be $E_f = \{(\mathbf{x}, z) : z \geq f(\mathbf{x})\} \subset \mathbb{R}^{n+1}$. Show that f is a convex function iff E_f is convex subset.
6. Give an example of a function which is strictly convex but whose second derivative is not everywhere > 0.
7. Show that x^2/y is convex on the upper half plane $(x, y) : y > 0$. * Show that if $f \in C^2(\mathbb{R})$ is convex then the function $yf(y^{-1}\mathbf{x})$ is convex on $(x, y) : y > 0$.
8. Given a family $L^\alpha(\mathbf{x})$ of affine functions indexed by $\alpha \in \mathbb{N}$, (or in fact an arbitrary index set) show that $f(\mathbf{x}) = \sup_\alpha L^\alpha(\mathbf{x})$ is convex. * Show that all C^1 convex functions arise in this way.
9. With L^α as in the previous question, show that the function $f(\mathbf{x}) = \inf_\alpha L^\alpha(\mathbf{x})$ is concave.
10. For A any real symmetric $n \times n$ matrix consider $\lambda(A) = \sup_{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| = 1} \mathbf{v} \cdot (A\mathbf{v})$. Use Lagrange multipliers to show that $\lambda(A)$ is the largest eigenvalue of A. * Also prove that λ is a convex function of A. (Assume the fact from analysis II that a continuous function on the sphere $\{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| = 1\}$ attains its supremum.)
11. The area A of a triangle with sides a, b, c is given by
$$A = \sqrt{[s(s-a)(s-b)(s-c)]}, \quad \text{where } s = \tfrac{1}{2}(a+b+c).$$
(i) Show that of all triangles of given perimeter $2s$, the triangle of largest area is equilateral. [*Hint:* Consider A^2.]
(ii) Find (in terms of the perimeter) the largest possible area of a right-angled triangle of given perimeter. [*Hint:* Do not use the expression A above.]
12. Find the Legendre transform of $f(x) = a^{-1}x^a, a > 1$ defined on $x > 0$, and deduce $xy \leq a^{-1}x^a + b^{-1}y^b$ for $a^{-1} + b^{-1} = 1$ (Young).
13. Find the Legendre transform of $f(\mathbf{x}) = \frac{1}{2}\sum_{ij} A_{ij} x^i x^j$ where A_{ij} is a positive symmetric matrix.
14. For an ideal gas, the internal energy $U = U(S, V)$ as a function of entropy and volume is
$$U = U_0 + \alpha n R T_0 \Big[\big(\frac{V_0}{V}\big)^{\frac{1}{\alpha}} e^{\frac{S - S_0}{\alpha n R}} - 1 \Big]$$
for some constants $U_0, T_0, V_0, S_0, \alpha, n, R$. Calculate the pressure and temperature (defined by $dU = TdS - pdV$), and verify that $pV = nRT$ (ideal gas equation of state). Calculate also the constant volume heat capacity $C_V = T\frac{\partial S}{\partial T}|_V$, and comment on the convexity of U as a function of S. Calculate the Helmholtz free energy $F = F(T, V)$ defined by $F(T, V) = \min_S (U(S, V) - TS)$.
15. For black body radiation the internal energy $U = U(S, V)$ as a function of entropy and volume is
$$U(S, V) = \Big(\frac{3S}{4}\Big)^{\frac{4}{3}} \Big(\frac{1}{CV}\Big)^{\frac{1}{3}}$$

where C is a constant. Calculate P, T as in the previous question and verify that the energy density (i.e. the internal energy per unit volume) is CT^4 and that the value of the pressure is $\frac{1}{3}$ of the energy density. Calculate the Helmholtz free energy $F = F(T, V)$ defined by $F(T, V) = \min_S(U(S, V) - TS)$, and show that it's value is $-\frac{1}{3}U$.

16. Show that the Euler-Lagrange equation of the functional

$$I[y] = \int_{x_1}^{x_2} f(y, y^{'})dx = 0, \quad y(x_1) = y_1 \text{ and } y(x_2) = y_2 \text{ fixed}$$

has the first integral $f(y, y^{'}) - y^{'}\frac{\partial}{\partial y'} f(y, y') =$ constant. The curve assumed by a uniform cable which is suspended between two points $(-a, b)$ and (a, b) minimises the potential energy

$$\int_{-a}^{a} y(1 + y'^2)^{\frac{1}{2}} dx$$

subject to the constraint that its length remains fixed,

$$\int_{-a}^{a} (1 + y'^2)^{\frac{1}{2}} dx = 2L,$$

where $L > a$. Show that the curve is a catenary

$$y - y_0 = c\cosh\left(\frac{x - x_0}{c}\right),$$

where c, x_0 and y_0 are constants. Find an equation for c, and show that it has a unique positive solution.

17. Write down the Euler-Lagrange equation for the functional

$$I[u] = \int_{-\infty}^{+\infty} \frac{1}{2}u'^2 + (1 - \cos u)dx$$

and find all solutions which satisfy $\lim_{x\to-\infty} u(x) = 0$ and $\lim_{x\to+\infty} u(x) = 2\pi$. Show that if $u \in C^1(\mathbb{R})$ satisfies $\lim_{x\to-\infty} u(x) = 0$ and $\lim_{x\to+\infty} u(x) = 2\pi$

$$I[u] = \frac{1}{2}\int_{-\infty}^{+\infty} (u' - 2\sin\frac{u}{2})^2\, dx + 8.$$

Deduce that a lower bound for $I[u]$ amongst such functions is 8, and give a *first order* differential equation which u must satisfy in order to realize this lower bound. Show that any solution of this first order equation solves the Euler-Lagrange equation you derived in the first part of the question. Give all the functions satisfying $I[u] = 8$.

18. * The brachistochrone problem leads to the study of the functional $I[y] = \int_0^X \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}dx$ for C^1 curves $y = y(x) > 0$ such that $y(0) = 0$ and $y(X) = Y > 0$. Make the change of variables $y = \phi^2$, and show that $J[\phi] = I[\phi^2] = \int_0^X (\phi^{-2} + 4\phi'^2)^{\frac{1}{2}}dx$. Show that the function $l(u, v) = (u^{-2} + 4v^2)^{\frac{1}{2}}$ is strictly convex on $\{(u, v) : u > 0\} \in \mathbb{R}^2$. (This can be used to prove the cycloid solution which we obtained as a solution of the Euler-Lagrange equation, which is only a necessary condition for a minimizer, actually does minimze I.) Write down the Euler-Lagrange equation for $J[\phi]$, solve for ϕ and show that the solutions are cycloids, as for the Euler-Lgrange equation for I.

19. Obtain the Euler-Lagrange equation for the function $x(t)$ that makes stationary the integral

$$\int_{t_1}^{t_2} f(t, x(t), \dot{x}(t), \ddot{x}(t))dt$$

for fixed values of both $x(t)$ and $\dot{x}(t)$ at both $t = t_1$ and $t = t_2$.
Find the function $x(t)$ with $x(1) = 1, \dot{x}(1) = -2, x(2) = \frac{1}{4}$ and $\dot{x}(2) = -\frac{1}{4}$, that minimises $\int_1^2 t^4[\ddot{x}(t)]^2 dt$, including a demonstration that it is a minimizer (not just a stationary point) for the integral.