Variational Principles: Example Sheet 2

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Comments/corrections to dmas2@cam.ac.uk. Sheet for supervisors will be available. The starred questions are not required for supervision.

1. Consider the problem of maximizing the area $\frac{1}{2}\int_0^{2\pi}(x\dot{y}-y\dot{x})dt$ enclosed by a *closed* curve of fixed length $l=\int_0^{2\pi}(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}dt$. Write down and solve the Euler-Lagrange equations for this constrained problem in parametric form.
2. Consider the problem of minimizing $I[\psi]=\int\big(\psi'^2+x^2\psi^2\big)dx$ amongst functions with $\int\psi^2dx=1$.
(i) Write down the corresponding Euler-Lagrange equation for this constrained problem.
(ii) Show that under the assumption $x\psi(x)^2\to 0$ as $x\to+\infty$ it is possible to write $I[\psi]=1+\int(\psi'+x\psi)^2dx$, and hence show that amongst such functions the minimum value of I is 1 and is attained on a function which should be given explicitly. Verify that this function satisfies the Euler-Lagrange equation you wrote down in (i), for an appropriate value of the Lagrange multiplier.
(iii) * Use the method of power series solutions to solve the Euler-Lagrange equation in (i), and comment on the relation with the minimizing function you obtained in (ii). (Here you may find it useful to rewrite the Euler-Lagrange equation as an equation for $f=e^{\frac{x^2}{2}}\psi(x)$).
3. Obtain the Euler-Lagrange equations associated to the functionals
(i) $I[u]=\int(\frac{1}{2}u_t^2-F(u_x))dxdt$,
(ii) $I[u]=\frac{1}{2}\int(u_t^2-c(u)^2u_x^2)dxdt$,
where $u=u(t,x)$ is a function on $\mathbb{R}^2$, where F and c are given smooth functions.
4. Obtain the Euler-Lagrange equations associated to the functionals
(i) $I[u]=\int(|\nabla u|^2+e^{2u})dxdy$,
where $u=u(x,y)$ is a function on $\mathbb{R}^2$, and
(ii) $I[u]=\int(\det Du)dxdy$,
where $u:\mathbb{R}^2\to\mathbb{R}^2$, and $\det Du$ means the Jacobian determinant. What is unusual about the second example?
5. Consider $I[y]=\int_{-1}^{+1}(xy')^2dx$ for $y(x)$ in the set S of C^1 functions such that $y(\pm1)=\pm1$. By considering $y_\epsilon(x)=\frac{\arctan x/\epsilon}{\arctan 1/\epsilon}$ show that $\inf_{y\in S}I[y]=0$. Show that this infimum is not attained in S.
6. Consider $I[y]=\int_{-1}^{1}(1-y_x^2)^2dx$ with $y=y(x)$ lying in the set S' of *piecewise* C^1 functions such that $y(\pm1)=1$. By considering $y(x)=|x|$ show that the $\min_{y\in S'}I[y]=0$. Does there exist a C^1 (not just piecewise C^1) function for which this value is attained?
7. The smooth functions $p(x),q(x)$ and $w(x)\geq 0$ are prescribed on $[a,b]$, with w not identically zero. Show that the following three conditions are equivalent for C^2 functions $y(x)$ satisfying $y(a)=0=y(b)$:
(i) y satisfies: $(py')'-qy=-\lambda wy$;
(ii) $I[u]=\int_a^b(pu'^2+qu^2)dx$ is stationary at $u=y$ amongst C^1 functions satisfying the boundary conditions and subject to the constraint $\int_a^b wu^2dx=$ constant;
(iii) $Q[u]=\int_a^b(pu'^2+qu^2)dx/\int_a^b wu^2dx$, is stationary amongst C^1 functions satisfying the boundary conditions at $u=y$. What is the value of $Q[y]$?
(Assume that y is not identically zero, and that $w>0$ in (a,b) so that so that the denominator $\int_a^b wy^2dx$ in (iii) is non-zero.)
8. Let $\mathbf{x}(t)\in\mathbb{R}^3$ be a curve which is constrained to lie on the sphere $S^2=\{\mathbf{x}:\|\mathbf{x}\|=1\}$. Use the Lagrange multiplier function formalism to obtain the following Euler-Lagrange equation
$$\ddot{\mathbf{x}}+\|\dot{\mathbf{x}}\|^2\mathbf{x}=0 \qquad (1)$$
for the problem of minimizing $I[\mathbf{x}]=\int\|\dot{\mathbf{x}}\|^2dt$ amongst curves satisfying the constraint

$\mathbf{x}(t) \in S^2$. Show that the solutions of the Euler-Lagrange equation lie on a plane through the origin (they are great circles.)

9. * As an alternative approach to (1), let θ, ϕ be standard angles given by spherical coordinates, and assume the curve on S^2 is given as $\phi = \phi(\theta)$. Show that the length integral is $l[\phi] = \int (1 + \sin^2\theta \phi'^2)^{\frac{1}{2}} d\theta$. Obtain the Euler-Lagrange equation associated to this functional, integrate it and show that the resulting solutions are great circles.
10. * Obtain (1) by considering variations of the curve $\mathbf{x}(t)$ of the form
$$\mathbf{x}^\epsilon(t) \equiv \frac{\mathbf{x}(t) + \epsilon\mathbf{z}(t)}{\|\mathbf{x}(t) + \epsilon\mathbf{z}(t)\|}$$
which lie on S^2 and requiring $\frac{d}{d\epsilon} I[\mathbf{x}^\epsilon] = 0$ at $\epsilon = 0$ for every smooth $\mathbf{z}(t)$.
11. For the brachistochrone problem, show that the minimum travel time between two points at the same level and a distance l apart is $(2\pi l/g)^{\frac{1}{2}}$ (for a bead moving on a wire under the action of gravity without friction. The acceleration due to gravity is g.)
12. For the brachistochrone problem, show that there is a unique arc of a cycloid (without a cusp) from the starting point $(0,0)$ to a point (X, Y) below the starting point.
13. In an optical medium filling the region $0 < y < h$, the speed of light is
$$c(y) = \frac{c_0}{(1 - ky)^{\frac{1}{2}}} \quad (0 < k < 1/h).$$
Show that the paths of light rays in the medium are parabolic. Show also that, if a ray enters the medium at $(-x_0, 0)$ and leaves it at $(x_0, 0)$, then
$$(kx_0)^2 = 4ky_0(1 - ky_0),$$
where y_0 $(< h)$ is the greatest value of y attained on the ray path.
14. * Hamilton's Principle is applicable also to the *relativistic* dynamics of a charged particle in an electromagnetic field. The appropriate choice of Lagrangian $L[t, \mathbf{x}(t), \dot{\mathbf{x}}(t)]$ is
$$L = -m_0 c^2 \gamma^{-1} - qA_0 - q\mathbf{v} \cdot \mathbf{A},$$
with the Lorentz factor $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$, and where $\mathbf{x}$ is the position and $\mathbf{v} = \dot{\mathbf{x}}(t)$ is the velocity of a particle of rest-mass m_0 and charge q in fields determined by a given scalar potential $A_0(\mathbf{x}, t)$ and a given vector potential $\mathbf{A}(\mathbf{x}, t)$. Verify that the Euler-Lagrange equations, with this choice of L, yield the equation of motion
$$\frac{d}{dt}(m_0 \gamma \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$
where the electric field $\mathbf{E} = \nabla A_0 - \frac{\partial \mathbf{A}}{\partial t}$ and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.
15. * With $\mathbf{E}$ and $\mathbf{B}$ as in the previous question, obtain the Euler-Lagrange equations associated to the functional $I[A] = \int (\mathbf{E}^2 - \mathbf{B}^2) dx dt$ (Maxwell equations).
16. For the length functional for curves in the plane $I[y] = \int_a^b (1 + y'^2)^{\frac{1}{2}} dx$, with $y(a) = \alpha$ and $y(b) = \beta$ show that the straight line $y = y_0(x)$ joining (a, α) to (b, β) solves the Euler-Lagrange equation. Compute the second variation of I at y_0 and show that it is positive.
17. For $I[y] = \int_a^b (y'^2 + y^4) dx$ with $y(a) = \alpha$, $y(b) = \beta$ find the Euler-Lagrange equation and the second variation. For the case $\alpha = 0 = \beta$ compute both the solution of the Euler-Lagrange equation and the second variation explicitly, and show that the second variation is strictly positive.
18. For $I[y] = \int_0^1 \left(\frac{1}{2} y'^2 + F(y)\right) dx$ with $y(0) = 0 = y(1)$. Assume that $F \in C^2(\mathbb{R})$ satisfies $F'(0) = 0$. Write down the associated Euler-Lagrange equation, and show that $y_0(x) = 0$ is a solution. Find the second variation. Give (i) a condition on $F''(0)$ which ensures that the second variation is positive, and (ii) a condition which ensures the second variation has at least one negative eigenvalue.