Complex Methods: Example Sheet 1

Part IB, Lent Term 2022
U. Sperhake

Comments are welcomed and may be sent to U.Sperhake@damtp.cam.ac.uk.
Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Cauchy–Riemann equations

1. (i) Where, if anywhere, in the complex plane are the following functions differentiable, and where are they analytic?

$$f(z) = \mathsf{Im}(z); \qquad f(z) = |z|^2; \qquad f(z) = \operatorname{sech} z := \frac{1}{\cosh z}.$$

(ii) Let $f(z) = z^5/|z|^4$, $z \neq 0$, $f(0) = 0$. Show that the real and imaginary parts of f satisfy the Cauchy–Riemann equations at $z = 0$, but that f is not differentiable at $z = 0$.

(iii) Given smooth functions $u(x, y)$, $v(x, y)$ we may define formally a function $g(z, \bar{z})$ by

$$g(z, \bar{z}) = u\big(\tfrac{1}{2}(z + \bar{z}), -\tfrac{1}{2}i(z - \bar{z})\big) + iv\big(\tfrac{1}{2}(z + \bar{z}), -\tfrac{1}{2}i(z - \bar{z})\big).$$

Using the chain rule and the Cauchy–Riemann equations, show that g is differentiable as a function of z if and only if $\partial g/\partial \bar{z} = 0$, and discuss how a dependence on $\bar{z}$ results in the nondifferentiability of f.

2. By calculation or inspection, find (as functions of z) complex analytic functions $f(z)$ whose real parts are the following:

(i) xy (ii) $\sin x \cosh y$ (iii) $\log(x^2 + y^2)$

(iv) $e^{y^2 - x^2} \cos 2xy$ (v) $\dfrac{y}{(x+1)^2 + y^2}$ (vi) $\tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$

Deduce that the above functions are harmonic on appropriately-chosen domains, which you should specify.

∗ **3.** By considering $w(z) = (\mathrm{i} + z)/(\mathrm{i} - z)$, show that $\phi(x, y) = \tan^{-1}\left(\dfrac{2x}{x^2 + y^2 - 1}\right)$ is harmonic.

4. Verify that the function $\phi(x, y) = e^x(x \cos y - y \sin y)$ is harmonic. Find its harmonic conjugate and, by considering $\nabla\phi$ or otherwise, determine the family of curves orthogonal to the family of curves $\phi(x, y) = \text{const}$.

Find an analytic function $f(z)$ such that $\mathsf{Re}(f) = \phi$. Can the expression $f(z) = \phi(z, 0)$ be used to determine $f(z)$ in general?

Branches of multi-valued functions

5. Show how the principal branch of $\log z$ can be used to define a branch of z^{i} which is single-valued and analytic on the domain $\mathcal{D} = \mathbb{C} \setminus (-\infty, 0]$. Evaluate i^{i} for this branch.

Show, using polar coordinates, that the branch of z^{i} defined above maps $\mathcal{D}$ onto an annulus which is covered infinitely often.

How would your answers change, if at all, for a different branch?

6. Exhibit three different branches of the function $z^{3/2}$ using at least two different branch cuts. Is it true that for $z, w \in \mathbb{C}$, $(zw)^{3/2} = z^{3/2}w^{3/2}$?

How many branch points does $[z(z+1)]^{1/3}$ have? Draw some different choices of possible branch cuts, both in the complex plane and on the Riemann sphere.

Repeat for $(z^2+1)^{1/2}$.

$*$ Repeat also for $[z(z+1)(z+2)]^{1/3}$ and $[z(z+1)(z+2)(z+3)]^{1/2}$.

7. Let $f(z) = (z^2-1)^{1/2}$, and consider two different branches of the function $f(z)$:

$f_1(z)$: branch cut $(-\infty,-1] \cup [1,\infty)$, with $f_1(x) = -\mathrm{i}\sqrt{1-x^2}$ for real $x \in (-1,1)$;

$f_2(z)$: branch cut $[-1,1]$, with $f_2(x) = +\sqrt{x^2-1}$ for real $x > 1$.

Find the limiting values of f_1 and f_2 above and below their respective branch cuts. Prove that f_1 is an even function, i.e., $f_1(z) = f_1(-z)$, and that f_2 is odd.

Conformal mappings

8. How does the disc $|z-1| < 1$ transform under the mapping $z \mapsto z^{-1}$?

Use the identity

$$\frac{z}{(z-1)^2} = \left(\frac{1}{1-z} - \frac{1}{2}\right)^2 - \frac{1}{4}$$

to show that the map $f(z) = z/(z-1)^2$ is a one-to-one conformal mapping of the disc $|z| < 1$ onto the domain $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

9. Consider the complex plane partitioned into eight open regions using as boundaries the real axis, the imaginary axis and the unit circle. Show that the map $z \mapsto (z-1)/(z+1)$ permutes these regions, and find the permutation.

What is the action of $z \mapsto (z-\mathrm{i})/(z+\mathrm{i})$ on these regions?

10. Find conformal mappings f_i of $\mathcal{U}_i$ onto $\mathcal{V}_i$ for each of the following cases. If the mapping is a composition of several functions, provide a sketch for each step. $\mathcal{D}$ denotes the unit disc $|z| < 1$.

(i) $\mathcal{U}_1$ is the angular sector $\{z : 0 < \arg z < \alpha\}$, $\mathcal{V}_1 = \{z : 0 < \mathsf{Im}(z) < 1\}$.

(ii) $\mathcal{U}_2 = \{z : \mathsf{Re}(z) < 0,\ -\frac{\pi}{2} < \mathsf{Im}(z) < \frac{\pi}{2}\}$, $\mathcal{V}_2$ is the quadrant $\{z : \mathsf{Re}(z) > 0,\ \mathsf{Im}(z) > 0\}$.

(iii) $\mathcal{U}_3 = \mathcal{D}$, $\mathcal{V}_3 = \mathcal{D} \setminus (-1, 0]$.

$*$ (iv) $\mathcal{U}_4$ is the open region bounded between two circles $\{z : |z| < 1,\ |z+\mathrm{i}| > \sqrt{2}\}$, $\mathcal{V}_4 = \mathcal{D}$.

Laplace's equation

11. Consider the half-strip $\mathcal{H} = \{z : -\frac{\pi}{2} < \mathsf{Re}(z) < \frac{\pi}{2},\ \mathsf{Im}(z) > 0\}$, the strip $\mathcal{S} = \{z : 0 < \mathsf{Im}(z) < \pi\}$, and the upper half-plane (UHP) $\{z : \mathsf{Im}(z) > 0\}$. Show that $g(z) = e^z$ maps $\mathcal{S}$ onto the UHP and that $h(z) = \sin z$ maps $\mathcal{H}$ onto the UHP.

Find a conformal map $f \colon \mathcal{H} \to \mathcal{S}$. Hence find a function $\phi(x,y)$ which is harmonic on the half-strip $\mathcal{H}$ with the following limiting values on its boundary $\partial\mathcal{H}$:

$$\phi(x,y) = \begin{cases} 0 & \text{on } \partial\mathcal{H} \text{ in the LHP } (x<0), \\ 1 & \text{on } \partial\mathcal{H} \text{ in the RHP } (x>0). \end{cases}$$

Give ϕ as a function of x and y. Is there only one such function?

$*$ **12.** Using conformal mapping(s), find a solution to Laplace's equation in the upper half-plane $\{(x, y) : y > 0\}$ with boundary conditions

$$\phi(x, 0) = \begin{cases} 1 & x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

[*Find a map f of the upper half-plane onto itself that makes the boundary conditions easier to deal with.*]