

# Complex Methods: Example Sheet 1

Part IB, Lent Term 2018

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

## Cauchy–Riemann equations

1. (i) Where, if anywhere, in the complex plane are the following functions differentiable, and where are they analytic?

$$\operatorname{Im} z; \quad |z|^2; \quad \operatorname{sech} z.$$

- (ii) Let  $f(z) = z^5/|z|^4$ ,  $z \neq 0$ ,  $f(0) = 0$ . Show that the real and imaginary parts of  $f$  satisfy the Cauchy–Riemann equations at  $z = 0$ , but that  $f$  is not differentiable at  $z = 0$ .

- (iii) Given smooth functions  $u(x, y)$ ,  $v(x, y)$  we may define formally a function  $g(z, \bar{z})$  by

$$g(z, \bar{z}) = u\left(\frac{1}{2}(z + \bar{z}), -\frac{1}{2}i(z - \bar{z})\right) + iv\left(\frac{1}{2}(z + \bar{z}), -\frac{1}{2}i(z - \bar{z})\right).$$

Using the chain rule and the Cauchy–Riemann equations, show that  $g$  is differentiable if and only if  $\partial g/\partial \bar{z} = 0$ , and explain the significance of this result.

2. By calculation or inspection, find (as functions of  $z$ ) complex analytic functions  $f(z)$  whose real parts are the following:

$$\begin{array}{lll} \text{(i)} \quad xy & \text{(ii)} \quad \sin x \cosh y & \text{(iii)} \quad \log(x^2 + y^2) \\ \text{(iv)} \quad e^{y^2 - x^2} \cos 2xy & \text{(v)} \quad \frac{y}{(x+1)^2 + y^2} & \text{(vi)} \quad \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right) \end{array}$$

Deduce that the above functions are harmonic on appropriately-chosen domains, which you should specify.

- \* 3. By considering  $w(z) = (i+z)/(i-z)$ , show that  $\phi(x, y) = \tan^{-1}\left(\frac{2x}{x^2 + y^2 - 1}\right)$  is harmonic.
4. Verify that the function  $\phi(x, y) = e^x(x \cos y - y \sin y)$  is harmonic. Find its harmonic conjugate and, by considering  $\nabla\phi$  or otherwise, determine the family of curves orthogonal to  $\phi(x, y) = c$  for a given constant  $c$ .
- Find an analytic function  $f(z)$  such that  $\operatorname{Re} f = \phi$ . Can the expression  $f(z) = \phi(z, 0)$  be used to determine  $f(z)$  in general?

## Branches of multi-valued functions

5. Show how the principal branch of  $\log z$  can be used to define a branch of  $z^i$  which is single-valued and analytic on the domain  $\mathcal{D} = \mathbb{C} \setminus (-\infty, 0]$ . Evaluate  $i^i$  for this branch.

Show, using polar coordinates, that the branch of  $z^i$  defined above maps  $\mathcal{D}$  onto an annulus which is covered infinitely often.

How would your answers change, if at all, for a different branch?

6. Exhibit three different branches of the function  $z^{3/2}$  using at least two different branch cuts. Is it true that for  $z, w \in \mathbb{C}$ ,  $(zw)^{3/2} = z^{3/2}w^{3/2}$ ?

How many branch points does  $[z(z+1)]^{1/3}$  have? Draw some different choices of possible branch cuts, both in the complex plane and on the Riemann sphere.

Repeat for  $(z^2 + 1)^{1/2}$ .

- \* Repeat also for  $[z(z+1)(z+2)]^{1/3}$  and  $[z(z+1)(z+2)(z+3)]^{1/2}$ .

7. Let  $f(z) = (z^2 - 1)^{1/2}$ , and consider two different branches of the function  $f(z)$ :

$f_1(z)$  : branch cut  $(-\infty, -1] \cup [1, \infty)$ , with  $f_1(x) = -i\sqrt{1 - x^2}$  for real  $x \in (-1, 1)$ ;

$f_2(z)$  : branch cut  $[-1, 1]$ , with  $f_2(x) = +\sqrt{x^2 - 1}$  for real  $x > 1$ .

Find the limiting values of  $f_1$  and  $f_2$  above and below their respective branch cuts. Prove that  $f_1$  is an even function, i.e.,  $f_1(z) = f_1(-z)$ , and that  $f_2$  is odd.

### Conformal mappings

8. How does the disc  $|z - 1| < 1$  transform under the mapping  $z \mapsto z^{-1}$ ?

Use the identity

$$\frac{z}{(z - 1)^2} = \left( \frac{1}{1 - z} - \frac{1}{2} \right)^2 - \frac{1}{4}$$

to show that the map  $f(z) = z/(z - 1)^2$  is a one-to-one conformal mapping of the disc  $|z| < 1$  onto the domain  $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ .

9. Consider the complex plane partitioned into eight open regions using as boundaries the real axis, the imaginary axis and the unit circle. Show that the map  $z \mapsto (z - 1)/(z + 1)$  permutes these regions, and find the permutation.

What is the action of  $z \mapsto (z - i)/(z + i)$  on these regions?

10. Find conformal mappings  $f_i$  of  $\mathcal{U}_i$  onto  $\mathcal{V}_i$  for each of the following cases. If the mapping is a composition of several functions, provide a sketch for each step.  $\mathcal{D}$  denotes the unit disc  $|z| < 1$ .

(i)  $\mathcal{U}_1$  is the angular sector  $\{z : 0 < \arg z < \alpha\}$ ,  $\mathcal{V}_1 = \{z : 0 < \operatorname{Im} z < 1\}$ .

(ii)  $\mathcal{U}_2 = \{z : \operatorname{Re} z < 0, -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\}$ ,  $\mathcal{V}_2$  is the quadrant  $\{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ .

(iii)  $\mathcal{U}_3 = \mathcal{D}$ ,  $\mathcal{V}_3 = \mathcal{D} \setminus (-1, 0]$ .

\* (iv)  $\mathcal{U}_4$  is the open region bounded between two circles  $\{z : |z| < 1, |z + i| > \sqrt{2}\}$ ,  $\mathcal{V}_4 = \mathcal{D}$ .

### Laplace's equation

11. Show that

$g(z) = e^z$  maps the strip  $\mathcal{S} = \{z : 0 < \operatorname{Im} z < \pi\}$  onto the UHP  $\{z : \operatorname{Im} z > 0\}$ ,

$h(z) = \sin z$  maps the half-strip  $\mathcal{H} = \{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im} z > 0\}$  onto the UHP.

Find a conformal map  $f: \mathcal{H} \rightarrow \mathcal{S}$ . Hence find a function  $\phi(x, y)$  which is harmonic on the half-strip  $\mathcal{H}$  with the following limiting values on its boundary  $\partial\mathcal{H}$ :

$$\phi(x, y) = \begin{cases} 0 & \text{on } \partial\mathcal{H} \text{ in the LHP } (x < 0), \\ 1 & \text{on } \partial\mathcal{H} \text{ in the RHP } (x > 0). \end{cases}$$

Give  $\phi$  as a function of  $x$  and  $y$ . Is there only one such function?

\* 12. Using conformal mapping(s), find a solution to Laplace's equation in the upper half-plane  $\{(x, y) : y > 0\}$  with boundary conditions

$$\phi(x, 0) = \begin{cases} 1 & x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

[Find a map  $f$  of the upper half-plane onto itself that makes the boundary conditions easier to deal with.]