Complex Methods: Example Sheet 2  
Part IB, Lent Term 2019  
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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk.  
Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Series expansions and singularities

1. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for \( \log(1 + z) \), etc.

   (i) \( z / \log(1 + z) \)   (ii) \( (\cos z)^{1/2} - 1 \)   (iii) \( \log(1 + e^z) \)   (iv) \( e^{ae^z} \)

   State the range of values of \( z \) for which each series converges. How would your answers differ if you assumed branches different from the principal branch?

2. Let \( a, b \) be complex constants, \( 0 < |a| < |b| \). Use partial fractions to find the Laurent expansions of \( 1 / ((z - a)(z - b)) \) about \( z = 0 \) in each of the regions \( |z| < |a| \), \( |a| < |z| < |b| \) and \( |z| > |b| \).

3. Find the first three terms of the Laurent expansion of \( f(z) = \cosec^2 z \) valid for \( 0 < |z| < \pi \).

   * Show that the function \( g(z) = f(z) - z^{-2} - (z + \pi)^{-2} - (z - \pi)^{-2} \) has only removably singularities in \( |z| < 2\pi \). Explain how to remove them to obtain a function \( G(z) \) analytic in that region. Find a Taylor Series for \( G(z) \) about the origin and explain why it must be convergent in \( |z| < 2\pi \). Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of \( f(z) \) valid for \( \pi < |z| < 2\pi \).

4. Show that if \( f(z) \) has a zero of order \( M \) and \( g(z) \) a zero of order \( N \) at \( z = z_0 \), then \( f(z)/g(z) \) has a zero of order \( M - N \) if \( M > N \), a removable singularity if \( M = N \), and a pole of order \( N - M \) if \( M < N \). Show also that \( 1/f(z) \) has a pole of order \( N \) if and only if \( f(z) \) has a zero of order \( N \).

5. Write down the location and type of each of the singularities of the following functions:

   (i) \( \frac{1}{z^2(z - 1)^2} \)   (ii) \( \tan z \)   (iii) \( \coth z \)   (iv) \( \frac{e^z - e}{(1 - z)^3} \)

   (v) \( \exp(\tan z) \)   (vi) \( \sinh \frac{z}{z^2 - 1} \)   (vii) \( \log(1 + e^z) \)   (viii) \( \tan(z^{-1}) \)

Integration and residues

6. Evaluate \( \int z \, dz \) along the straight line from \(-1\) to \(+1\), and along the semicircular contour in the upper half-plane between the same two points; and evaluate \( \oint \gamma \, \bar{z} \, dz \) when \( \gamma \) is the circle \( |z| = 1 \), and when \( \gamma \) is the circle \( |z - 1| = 1 \).

7. (i) Show that if \( f(z) \) and \( g(z) \) are analytic, and \( g \) has a simple zero at \( z = z_0 \), the residue of \( f(z)/g(z) \) at \( z = z_0 \) is \( f(z_0)/g'(z_0) \). In particular, show that \( f(z)/g(z) \) has residue \( f(z_0) \).

   (ii) Prove the formula for the residue of a function \( f(z) \) that has a pole of order \( N \) at \( z = z_0 \):

   \[
   \lim_{z \to z_0} \left\{ \frac{1}{(N - 1)!} \frac{d^{N-1}}{dz^{N-1}} ((z - z_0)^N f(z)) \right\}.
   \]

   (iii) Find the residues of the poles in question 5.
8. Evaluate, using Cauchy’s theorem or the residue theorem,

\[ \int_{\gamma_1} \frac{dz}{1 + z^2} \quad (i) \quad \int_{\gamma_2} \frac{dz}{1 + z^2} \quad (ii) \quad \int_{\gamma_3} \frac{e^z \cot zdz}{1 + z^2} \quad (iii) \quad \int_{\gamma_4} \frac{z^{3/2}dz}{1 + z} \quad \text{ (iv)} \]

where \( \gamma_1 \) is the elliptical contour \( (\text{Re } z)^2 + 4(\text{Im } z)^2 = 1 \), \( \gamma_2 \) is the circle \( |z| = \sqrt{2} \), \( \gamma_3 \) is the circle \( |z - (2 + i)| = \sqrt{2} \) and \( \gamma_4 \) is the circle \( |z| = 2 \), all traversed anti-clockwise.

9. By integrating around a keyhole contour, show that

\[ \int_0^{2\pi} \cos n\theta \frac{d\theta}{1 - 2a \cos \theta + a^2} \]

where \( a \) is real, \( a > 1 \), and \( n \) is a non-negative integer.

* Obtain the same result using Cauchy’s integral formula instead of the residue theorem.

**The calculus of residues**

10. Evaluate \( \int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2} \).

11. By integrating around a keyhole contour, show that

\[ \int_0^\infty \frac{x^{p-1} dx}{1 + x} = -\frac{\pi}{\sin \pi a} \quad (0 < a < 1). \]

Explain why the given restrictions on the value of \( a \) are necessary.

* 12. By integrating around a contour involving the real axis and the line \( z = re^{2\pi i/n}, \) evaluate \( \int_0^\infty \frac{dx}{(1 + x^n)}, \) \( n \geq 2 \). Check (by change of variable) that your answer agrees with that of the previous question.

13. Establish the following:

\[ \int_0^\infty \frac{\cos x}{(1 + x^2)^3} \, dx = \frac{7\pi}{16e} \quad (i) \quad \int_0^\infty x^2 \sech x \, dx = \frac{\pi^3}{8} \frac{\gamma_4}{1 + z} \quad (ii) \quad \int_0^\infty \log x \frac{dx}{1 + x^2} = 0 \quad \text{ (iii)} \quad \int_0^\infty \sin^2 x \frac{dx}{x^2} = \frac{\pi}{2} \]

\[ \text{For part (ii), use a rectangular contour. For part (iii), integrate } (\log z)^2/(1 + z^2) \text{ around a keyhole, or } (\log z)/(1 + z^2) \text{ along the real axis (or both). What goes wrong with } (\log z)/(1 + z^2) \text{ around a keyhole?} \]

* 14. Let \( P(z) \) be a non-constant polynomial. Consider the contour integral \( I = \oint_{\gamma} \frac{P'(z)}{P(z)} \, dz \). Show that, if \( \gamma \) is a contour that encloses no zeros of \( P \), then \( I = 0 \).

Evaluate the limit of \( I \) as \( R \to \infty \), where \( \gamma \) is the circle \( |z| = R \), and deduce that \( P \) has at least one zero in the complex plane.

15. By considering the integral of \((\cot z)/(z^2 + \pi^2 a^2)\) around a suitable large contour, prove that

\[ \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a \]

provided that \( ia \) is not an integer. By considering a similar integral prove also that, if \( a \) is not an integer,

\[ \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}. \]

Find an expression for \( \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \) and take the limit as \( a \to 0 \) to deduce the value of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).