Complex Methods: Example Sheet 2
Part IB, Lent Term 2020
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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. 
Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Series expansions and singularities

1. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for \( \log(1+z) \), etc.
   
   \[
   (\text{i}) \frac{z}{\log(1+z)} \quad (\text{ii}) (\cos z)^{1/2} - 1 \quad (\text{iii}) \log(1+e^z) \quad (\text{iv}) e^{az}
   \]
   
   State the range of values of \( z \) for which each series converges.
   
   How would your answers differ if you assumed branches different from the principal branch?

2. Let \( a, b \) be complex constants, \( 0 < |a| < |b| \). Use partial fractions to find the Laurent expansions of \( 1/\{(z-a)(z-b)\} \) about \( z = 0 \) in each of the regions \( |z| < |a|, |a| < |z| < |b| \) and \( |z| > |b| \).

3. Find the first three terms of the Laurent expansion of \( f(z) = \cosec^2 z \) valid for \( 0 < |z| < \pi \). 
   
   * Show that the function \( g(z) = f(z) - z^{-2} - (z+\pi)^{-2} - (z-\pi)^{-2} \) has only removable singularities in \( |z| < 2\pi \). Explain how to remove them to obtain a function \( G(z) \) analytic in that region. Find a Taylor Series for \( G(z) \) about the origin and explain why it must be convergent in \( |z| < 2\pi \). Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of \( f(z) \) valid for \( \pi < |z| < 2\pi \).

4. Show that if \( f(z) \) has a zero of order \( M \) and \( g(z) \) a zero of order \( N \) at \( z = z_0 \), then \( f(z)/g(z) \) has a zero of order \( M-N \) if \( M > N \), a removable singularity if \( M = N \), and a pole of order \( N-M \) if \( M < N \). Show also that \( 1/f(z) \) has a pole of order \( N \) if and only if \( f(z) \) has a zero of order \( N \).

5. Write down the location and type of each of the singularities of the following functions:
   
   \[
   (\text{i}) \frac{1}{z^n(z-1)^2} \quad (\text{ii}) \tan z \quad (\text{iii}) z \coth z \quad (\text{iv}) \frac{e^z - e}{(1-z)^3} \\
   (\text{v}) \exp(\tan z) \quad (\text{vi}) \sinh \frac{z}{z^2 - 1} \quad (\text{vii}) \log(1+e^z) \quad (\text{viii}) \tan(z^{-1})
   \]

Integration and residues

6. Evaluate \( \int z \, dz \) along the straight line from \(-1\) to \(+1\), and along the semicircular contour in the upper half-plane between the same two points; and evaluate \( \oint_{\gamma} \bar{z} \, dz \) when \( \gamma \) is the circle \( |z| = 1 \), and when \( \gamma \) is the circle \( |z-1| = 1 \).

7. (i) Show that if \( f(z) \) and \( g(z) \) are analytic, and \( g \) has a simple zero at \( z = z_0 \), the residue of \( f(z)/g(z) \) at \( z = z_0 \) is \( f(z_0)/g'(z_0) \). In particular, show that \( f(z)/g(z) \) has residue \( f(z_0) \).
   
   (ii) Prove the formula for the residue of a function \( f(z) \) that has a pole of order \( N \) at \( z = z_0 \):
   
   \[
   \lim_{z \to z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} \left( (z - z_0)^N f(z) \right) \right\}.
   \]
   
   (iii) Find the residues of the poles in question 5.
8. Evaluate, using Cauchy’s theorem or the residue theorem,

\[
\begin{align*}
(i) & \int_{\gamma_1} \frac{dz}{1+z^2} \\
(ii) & \int_{\gamma_2} \frac{dz}{1+z^2} \\
(iii) & \int_{\gamma_3} \frac{e^z \cot zdz}{1+z^2} \\
(iv) & \int_{\gamma_4} \frac{z^3e^{1/z} dz}{1+z}
\end{align*}
\]

where \(\gamma_1\) is the elliptical contour (\(\Re z)^2 + 4(\Im z)^2 = 1\), \(\gamma_2\) is the circle \(|z| = \sqrt{2}\), \(\gamma_3\) is the circle \(|z - (2 + i)| = \sqrt{2}\) and \(\gamma_4\) is the circle \(|z| = 2\), all traversed anti-clockwise.

9. By integrating the function \(z^n(z - a)^{-1}(z - a^{-1})^{-1}\) around the unit circle and applying the residue theorem, evaluate

\[\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a \cos \theta + a^2} d\theta\]

where \(a\) is real, \(a > 1\), and \(n\) is a non-negative integer.

* Obtain the same result using Cauchy’s integral formula instead of the residue theorem.

**The calculus of residues**

10. Evaluate \(\int_{-\infty}^{\infty} \frac{dx}{1+x+x^2}\).

11. By integrating around a keyhole contour, show that

\[\int_0^{\infty} \frac{x^{p-1} dx}{1+x} = \frac{\pi}{\sin \pi a} \quad (0 < a < 1).\]

Explain why the given restrictions on the value of \(a\) are necessary.

* 12. By integrating around a contour involving the real axis and the line \(z = re^{2\pi i/n}\), evaluate \(\int_0^{\infty} \frac{dx}{(1+x^n)}, \ n \geq 2\). Check (by change of variable) that your answer agrees with that of the previous question.

13. Establish the following:

\[
\begin{align*}
(i) & \int_0^{\infty} \frac{\cos x}{(1+x^2)^3} dx = \frac{7\pi}{16e} \\
(ii) & \int_0^{\infty} x^2 \sech x dx = \frac{\pi^3}{8} \\
(iii) & \int_0^{\infty} \log x \frac{dx}{1+x^2} = 0 \\
(iv) & \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}
\end{align*}
\]

[For part (ii), use a rectangular contour. For part (iii), integrate \((\log z)^2/(1+z^2)\) around a keyhole, or \((\log z)/(1+z^2)\) along the real axis (or both). What goes wrong with \((\log z)/(1+z^2)\) around a keyhole?]

* 14. Let \(P(z)\) be a non-constant polynomial. Consider the contour integral \(I = \oint_{\gamma} \frac{P'(z)}{P(z)} dz\). Show that, if \(\gamma\) is a contour that encloses no zeros of \(P\), then \(I = 0\).

Evaluate the limit of \(I\) as \(R \to \infty\), where \(\gamma\) is the circle \(|z| = R\), and deduce that \(P\) has at least one zero in the complex plane.

15. By considering the integral of \((\cot z)/(z^2 + \pi^2 a^2)\) around a suitable large contour, prove that

\[\sum_{n=\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a\]

provided that \(ia\) is not an integer. By considering a similar integral prove also that, if \(a\) is not an integer,

\[\sum_{n=\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.
\]

Find an expression for \(\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}\) and take the limit as \(a \to 0\) to deduce the value of \(\sum_{n=1}^{\infty} \frac{1}{n^2}\).