Taylor and Laurent series

1. Find the first two non-vanishing coefficients in the Taylor expansion about the origin of the following functions, assuming principal branches for (i), (ii) and (iii). Where appropriate, you may make use of standard series expansions for \( \log(1 + z) \), etc.

   (i) \( z/\log(1 + z) \);
   (ii) \( \sqrt{\cos z - 1} \);
   (iii) \( \log(1 + e^z) \);
   (iv) \( e^{z^2} \).

State the range of values of \( z \) for which each series converges.
In (i), (ii) and (iii), how would your answer differ if you assumed branches different from the principal branch?

2. Use partial fractions to find the Laurent expansions of \( \frac{1}{(z - a)(z - b)} \) about \( z = 0 \), where \( 0 < |a| < |b| \), in each of the regions \( |z| < |a|, |a| < |z| < |b| \) and \( |b| < |z| \).

3. Find the first three terms of the Laurent expansion of \( f(z) = \frac{1}{\sin^2 z} \) valid for \( 0 < |z| < \pi \).

*(*) By considering the function

\[
h(z) = f(z) - \frac{1}{z^2} - \frac{1}{(z + \pi)^2} - \frac{1}{(z - \pi)^2}.
\]

find the three non-zero central terms of the Laurent expansion of \( f(z) \) valid for \( \pi < |z| < 2\pi \).

4. Write down the positions in the complex plane and the types of the singularities of the following functions:

   (i) \( \frac{1}{z^3(z - 1)^2} \);
   (ii) \( \tan z \);
   (iii) \( z \coth z \);
   (iv) \( \frac{e^z - e}{(1 - z)^3} \);
   (v) \( \exp(\tan z) \);
   (vi) \( \log(1 + e^z) \);
   (vii) \( \tan(z^{-1}) \).

Integration and residue calculus

5. Evaluate \( \oint_C z \, dz \) and \( \oint_C z^{\frac{1}{2}} \, dz \) (use the principal branch of \( z^{\frac{1}{2}} \)) in the two cases

   (i) \( C \) is the circle \( |z| = 1 \), (ii) \( C \) is the circle \( |z - 1| = 1 \).

6. Evaluate, using Cauchy’s theorem or the residue theorem:

   (i) \( \oint_C \frac{dz}{1 + z^2} \) where \( C \) is the ellipse \( x^2 + 4y^2 = 1 \);
   (ii) \( \oint_C \frac{dz}{1 + z^2} \) where \( C \) is the circle \( x^2 + y^2 = 2 \);
   (iii) \( \oint_C \frac{e^{z \cos z} \, dz}{(1 + z^2) \sin z} \) where \( C \) is the circle \( |z - (2 + i)| = \sqrt{2} \).

7. Evaluate \( \oint_C \frac{z^{3} e^{1/z} \, dz}{1 + z} \), where \( C \) is the circle \( |z| = 2 \).
8. (i) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$ by closing the contour in the upper half-plane.

How does the calculation differ if you close in the lower half-plane?

(ii) Evaluate $\lim_{R \to \infty} \int_{-R}^{R} \frac{x \ dx}{1 + x + x^2}$. Why is the limit here rather than just the integral $\int_{-\infty}^{\infty}$?

(iii) Evaluate $\int_{-\infty}^{\infty} e^{ikx} \frac{dx}{1 + x^2}$ for $k > 0$ and for $k < 0$.

9. By integrating round a key-hole contour, show that

$$\int_{0}^{\infty} \frac{x^{-a} \ dx}{1 + x} = \frac{\pi}{\sin(\pi a)} \quad (0 < a < 1).$$

Explain why the given restrictions on the value of $a$ are necessary.

10. By integrating round a contour involving the real axis and the line $z = re^{2\pi i/n}$, evaluate

$$\int_{0}^{\infty} \frac{dx}{1 + x^n} \quad (n \geq 2).$$

Check (by change of variable) that your answer agrees with that of the previous question.

11. Show that

$$\int_{0}^{2\pi} \frac{d\theta}{a + \cos \theta} = \frac{2\pi}{\sqrt{a^2 - 1}}, \quad a > 1.$$

Hence derive that

$$\int_{0}^{2\pi} \frac{d\theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} = \frac{2\pi}{ab}, \quad (0 < a < b).$$

12. Establish the following:

(i) $\int_{0}^{\infty} \frac{\cos x}{(1 + x^2)^{3/2}} \ dx = \frac{7\pi}{16}$;

(ii) $\int_{0}^{\infty} \frac{\sin^2 x}{x^2} \ dx = \frac{\pi}{2}$;

(iii) $\int_{0}^{\infty} \log x \ dx = 0$.

[For part (iii), integrate $\frac{(\log z)^2}{1 + z^2}$ around a keyhole, or $\frac{\log z}{1 + z^2}$ along the real axis (or both). What goes wrong if you integrate $\log z$ around a keyhole?]

13. Let $P(z)$ be a non-constant polynomial. Consider the contour integral

$$I = \oint_{C} \frac{P'(z)}{P(z)} \ dz.$$

Show that, if $C$ is a contour that encloses no zeros of $P$, then $I = 0$. Evaluate $\lim_{R \to \infty} I$, where $C$ is the circle $|z| = R$, and deduce that $P$ has at least one zero in the complex plane.

14. By considering the integral of $f(z) = \frac{\cot z}{z^2 + \pi^2 a^2}$ around a suitable large contour, prove that (provided $ia$ is not an integer)

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a).$$

By considering a similar integral, prove also that, if $a$ is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n + a)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

Deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ from the equality $\sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} = \left[ \frac{\pi}{a} \coth(\pi a) - \frac{1}{a^2} \right]$ justifying the limit pass $a \to 0$. 
