Complex Methods: Example Sheet 2

Part IB, Lent Term 2018
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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Series expansions and singularities

1. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for $\log(1+z)$, etc.

$$\text{(i)}\ \ z/\log(1+z) \qquad \text{(ii)}\ \ (\cos z)^{1/2}-1 \qquad \text{(iii)}\ \ \log(1+e^z) \qquad \text{(iv)}\ \ e^{e^z}$$

State the range of values of z for which each series converges.

How would your answers differ if you assumed branches different from the principal branch?

2. Let a, b be complex constants with $0 < |a| < |b|$. Use partial fractions to find the Laurent expansions of $1/\{(z-a)(z-b)\}$ about $z=0$ in each of the regions $|z|<|a|$, $|a|<|z|<|b|$ and $|z|>|b|$.

3. Find the first three terms of the Laurent expansion of $f(z) = \operatorname{cosec}^2 z$ valid for $0<|z|<\pi$.

$*$ Show that the function $g(z) = f(z) - z^{-2} - (z+\pi)^{-2} - (z-\pi)^{-2}$ has only removable singularities in $|z|<2\pi$. Explain how to remove them to obtain a function $G(z)$ analytic in that region. Find a Taylor Series for $G(z)$ about the origin and explain why it must be convergent in $|z|<2\pi$. Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of $f(z)$ valid for $\pi<|z|<2\pi$.

4. Show that $f(z)$ has a zero of order N at $z=z_0$ if and only if $1/f(z)$ has a pole of order N there.

5. Write down the location and type of each of the singularities of the following functions:

$$\text{(i)}\ \ \frac{1}{z^3(z-1)^2} \qquad \text{(ii)}\ \ \tan z \qquad \text{(iii)}\ \ z\coth z \qquad \text{(iv)}\ \ \frac{e^z-e}{(1-z)^3}$$

$$\text{(v)}\ \ \exp(\tan z) \qquad \text{(vi)}\ \ \sinh\frac{z}{z^2-1} \qquad \text{(vii)}\ \ \log(1+e^z) \qquad \text{(viii)}\ \ \tan(z^{-1})$$

Integration and residues

6. Evaluate $\int z\,\mathrm{d}z$ along the straight line from -1 to $+1$, and along the semicircular contour in the upper half-plane between the same two points; and evaluate $\oint_\gamma \bar{z}\,\mathrm{d}z$ when γ is the circle $|z|=1$, and when γ is the circle $|z-1|=1$.

7. (i) Show that if $f(z)$ is analytic, then the residue of $f(z)/(z-z_0)$ at $z=z_0$ is $f(z_0)$.

(ii) Show that if $f(z)$ is analytic and non-zero but $g(z)$ has a simple zero at $z=z_0$, the residue of $f(z)/g(z)$ at $z=z_0$ is $f(z_0)/g'(z_0)$.

(iii) Prove the formula for the residue of a function $f(z)$ that has a pole of order N at $z=z_0$:

$$\lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\big((z-z_0)^N f(z)\big)\right\}.$$

(iv) Find the residues of the poles in question 5.

8. Evaluate, using Cauchy's theorem or the residue theorem,

$$\text{(i)} \oint_{\gamma_1} \frac{\mathrm{d}z}{1+z^2} \qquad \text{(ii)} \oint_{\gamma_2} \frac{\mathrm{d}z}{1+z^2} \qquad \text{(iii)} \oint_{\gamma_3} \frac{e^z \cot z \,\mathrm{d}z}{1+z^2} \qquad * \text{ (iv)} \oint_{\gamma_4} \frac{z^3 e^{1/z}\,\mathrm{d}z}{1+z}$$

where γ_1 is the elliptical contour $(\operatorname{Re} z)^2 + 4(\operatorname{Im} z)^2 = 1$, γ_2 is the circle $|z| = \sqrt{2}$, γ_3 is the circle $|z-(2+i)| = \sqrt{2}$ and γ_4 is the circle $|z| = 2$, all traversed anti-clockwise.

9. By integrating the function $z^n(z-a)^{-1}(z-a^{-1})^{-1}$ around the unit circle and applying the residue theorem, evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta$$

where a is real, $a > 1$, and n is a non-negative integer.

$*$ Obtain the same result using Cauchy's integral formula instead of the residue theorem.

The calculus of residues

10. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x+x^2}$.

11. By integrating around a keyhole contour, show that

$$\int_0^\infty \frac{x^{a-1}\,\mathrm{d}x}{1+x} = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1).$$

Explain why the given restrictions on the value of a are necessary.

$*$ **12.** By integrating around a contour involving the real axis and the line $z = re^{2\pi i/n}$, evaluate $\int_0^\infty \mathrm{d}x/(1+x^n)$, $n \geqslant 2$. Check (by change of variable) that your answer agrees with that of the previous question.

13. Establish the following:

$$\text{(i)} \int_0^\infty \frac{\cos x}{(1+x^2)^3}\,\mathrm{d}x = \frac{7\pi}{16e} \qquad \text{(ii)} \int_0^\infty \operatorname{sech} x\,\mathrm{d}x = \frac{\pi}{2}$$

$$\text{(iii)} \int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x = 0 \qquad * \text{ (iv)} \int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}$$

[*For part (ii), use a rectangular contour. For part (iii), integrate* $(\log z)^2/(1+z^2)$ *around a keyhole, or* $\log z/(1+z^2)$ *along the real axis (or both). What goes wrong with* $\log z/(1+z^2)$ *around a keyhole?*]

$*$ **14.** Let $P(z)$ be a non-constant polynomial. Consider the contour integral $I = \oint_\gamma \big(P'(z)/P(z)\big)\,\mathrm{d}z$. Show that, if γ is a contour that encloses no zeros of P, then $I = 0$.

Evaluate the limit of I as $R \to \infty$, where γ is the circle $|z| = R$, and deduce that P has at least one zero in the complex plane.

15. By considering the integral of $f(z) = \cot z/(z^2 + \pi^2 a^2)$ around a suitable large contour, prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth \pi a$$

provided that ia is not an integer. By considering a similar integral prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

Find an expression for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$ and take the limit as $a \to 0$ to deduce the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.