Complex Methods: Example Sheet 2
Part IB, Lent Term 2024
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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Series expansions and singularities

1. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for \(\log(1 + z)\), etc.

   (i) \(z/\log(1 + z)\)  (ii) \((\cos z)^{1/2} - 1\)  (iii) \(\log(1 + e^z)\)  (iv) \(e^{e^z}\)

   State the range of values of \(z\) for which each series converges.

   How would your answers differ if you assumed branches different from the principal branch?

2. Let \(a, b\) be complex constants, \(0 < |a| < |b|\). Use partial fractions to find the Laurent expansions of \(1/\{(z - a)(z - b)\}\) about \(z = 0\) in each of the regions \(|z| < |a|, |a| < |z| < |b|\) and \(|z| > |b|\).

3. Find the first three terms of the Laurent expansion of \(f(z) = \frac{1}{\sin^n z}\) valid for \(0 < |z| < \pi\).

   * Show that the function \(g(z) = f(z) - z^{-2} - (z + \pi)^{-2} - (z - \pi)^{-2}\) has only removable singularities in \(|z| < 2\pi\).

   Explain how to remove them to obtain a function \(G(z)\) analytic in that region. Find a Taylor Series for \(G(z)\) about the origin and explain why it must be convergent in \(|z| < 2\pi\).

   Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of \(f(z)\) valid for \(\pi < |z| < 2\pi\).

4. Show that if \(f(z)\) has a zero of order \(M\) and \(g(z)\) a zero of order \(N\) at \(z = z_0\), then \(f(z)/g(z)\) has a zero of order \(M - N\) if \(M > N\), a removable singularity if \(M = N\), and a pole of order \(N - M\) if \(M < N\). Show also that \(1/f(z)\) has a pole of order \(N\) if and only if \(f(z)\) has a zero of order \(N\).

5. Write down the location and type of each of the singularities of the following functions:

   (i) \(\frac{1}{z^n(z - 1)^2}\)  (ii) \(\tan z\)  (iii) \(z \coth z\)  (iv) \(\frac{e^z - e}{(1 - z)^3}\)

   (v) \(\exp(\tan z)\)  (vi) \(\sinh\frac{z}{z^2 - 1}\)  (vii) \(\log(1 + e^z)\)  (viii) \(\tan(z^{-1})\)

Integration and residues

6. Evaluate \(\int z\,dz\) along the straight line from \(-1\) to \(+1\), and along the semicircular contour in the upper half-plane between the same two points; and evaluate \(\oint_{\gamma} z\,dz\) when \(\gamma\) is the circle \(|z| = 1\), and when \(\gamma\) is the circle \(|z - 1| = 1\).

7. (i) Show that if \(f(z)\) and \(g(z)\) are analytic, and \(g\) has a simple zero at \(z = z_0\), the residue of \(f(z)/g(z)\) at \(z = z_0\) is \(f(z_0)/g'(z_0)\). In particular, show that \(f(z)/(z - z_0)\) has residue \(f(z_0)\).

   (ii) Prove the formula for the residue of a function \(f(z)\) that has a pole of order \(N\) at \(z = z_0\):

   \[
   \lim_{z \to z_0} \left\{ \frac{1}{(N - 1)!} \frac{d^{N-1}}{dz^{N-1}} \left( (z - z_0)^N f(z) \right) \right\}.
   \]

   (iii) Find the residues of the poles in question 5.
8. Evaluate, using Cauchy’s theorem or the residue theorem,

\[ \begin{align*}
\text{(i)} & \int_{\gamma_1} \frac{dz}{1 + z^2} \\
\text{(ii)} & \int_{\gamma_2} \frac{dz}{1 + z^2} \\
\text{(iii)} & \int_{\gamma_3} \frac{e^{z} \cot z \, dz}{1 + z^2} \quad \ast \\
\text{(iv)} & \int_{\gamma_4} \frac{z^{3} e^{1/z} \, dz}{1 + z}
\end{align*} \]

where \( x = \Re(z), \ y = \Im(z), \ \gamma_1 \) is the elliptical contour \( x^2 + 4y^2 = 1, \ \gamma_2 \) is the circle \( |z| = \sqrt{2}, \ \gamma_3 \) is the circle \( |z - (2 + i)| = \sqrt{2} \) and \( \gamma_4 \) is the circle \( |z| = 2 \), all traversed anti-clockwise.

9. By integrating the function \( z^n(z - a)^{-1}(z - a^{-1})^{-1} \) around the unit circle and applying the residue theorem, evaluate

\[ \int_{0}^{2\pi} \frac{\cos n\theta}{1 - 2a \cos \theta + a^2} \, d\theta \]

where \( a \) is real, \( a > 1 \), and \( n \) is a non-negative integer.

\[ \ast \] Obtain the same result using Cauchy’s integral formula instead of the residue theorem.

**The calculus of residues**

10. Evaluate \( \int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2} \).

11. By integrating around a keyhole contour, show that

\[ \int_{0}^{\infty} \frac{x^{a-1} \, dx}{1 + x} = \frac{\pi}{\sin \pi a} \quad (0 < a < 1). \]

Explain why the given restrictions on the value of \( a \) are necessary.

\[ \ast \] 12. By integrating around a contour involving the real axis and the line \( z = re^{2\pi i/n} \), evaluate \( \int_{0}^{\infty} dx/(1 + x^n) \), \( n \geq 2 \). Check (by change of variable) that your answer agrees with that of the previous question.

13. Establish the following:

\[ \begin{align*}
\text{(i)} & \int_{0}^{\infty} \frac{\cos x}{(1 + x^2)^3} \, dx = \frac{7\pi}{16} \\
\text{(ii)} & \int_{0}^{\infty} \frac{x^{2}}{\cosh x} \, dx = \frac{\pi^3}{8} \\
\text{(iii)} & \int_{0}^{\infty} \frac{\log x}{1 + x^2} \, dx = 0 \quad \ast \\
\text{(iv)} & \int_{0}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2}
\end{align*} \]

[For part (ii), use a rectangular contour. For part (iii), integrate \((\log z)^2/(1 + z^2)\) around a keyhole, or \((\log z)/(1 + z^2)\) along the real axis (or both). What goes wrong with \((\log z)/(1 + z^2)\) around a keyhole?]

\[ \ast \] 14. Let \( P(z) \) be a non-constant polynomial. Consider the contour integral \( I = \oint_{\gamma} \frac{P'(z)}{P(z)} \, dz \). Show that, if \( \gamma \) is a contour that encloses no zeros of \( P \), then \( I = 0 \).

Evaluate the limit of \( I \) as \( R \to \infty \), where \( \gamma \) is the circle \( |z| = R \), and deduce that \( P \) has at least one zero in the complex plane.

15. By considering the integral of \((\cot z)/(z^2 + \pi^2 a^2)\) around a suitable large contour, prove that

\[ \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a \]

provided that \( ia \) is not an integer. By considering a similar integral prove also that, if \( a \) is not an integer,

\[ \sum_{n=-\infty}^{\infty} \frac{1}{(n + a)^2} = \frac{\pi^2}{\sin^2 \pi a}. \]

Find an expression for \( \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \) and take the limit as \( a \to 0 \) to deduce the value of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).