

# Complex Methods: Example Sheet 2

Part IB, Lent Term 2025

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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

## Series expansions and singularities

1. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for  $\log(1+z)$ , etc.

(i)  $z/\log(1+z)$       (ii)  $(\cos z)^{1/2} - 1$       (iii)  $\log(1+e^z)$       (iv)  $e^{e^z}$

State the range of values of  $z$  for which each series converges.

How would your answers differ if you assumed branches different from the principal branch?

2. Let  $a, b$  be complex constants,  $0 < |a| < |b|$ . Use partial fractions to find the Laurent expansions of  $1/\{(z-a)(z-b)\}$  about  $z=0$  in each of the regions  $|z| < |a|$ ,  $|a| < |z| < |b|$  and  $|z| > |b|$ .
3. Find the first three terms of the Laurent expansion of  $f(z) = \frac{1}{\sin^2 z}$  valid for  $0 < |z| < \pi$ .
- \* Show that the function  $g(z) = f(z) - z^{-2} - (z+\pi)^{-2} - (z-\pi)^{-2}$  has only removable singularities in  $|z| < 2\pi$ . Explain how to remove them to obtain a function  $G(z)$  analytic in that region. Find a Taylor Series for  $G(z)$  about the origin and explain why it must be convergent in  $|z| < 2\pi$ . Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of  $f(z)$  valid for  $\pi < |z| < 2\pi$ .
4. Show that if  $f(z)$  has a zero of order  $M$  and  $g(z)$  a zero of order  $N$  at  $z = z_0$ , then  $f(z)/g(z)$  has a zero of order  $M - N$  if  $M > N$ , a removable singularity if  $M = N$ , and a pole of order  $N - M$  if  $M < N$ . Show also that  $1/f(z)$  has a pole of order  $N$  if and only if  $f(z)$  has a zero of order  $N$ .
5. Write down the location and type of each of the singularities of the following functions:

(i)  $\frac{1}{z^3(z-1)^2}$       (ii)  $\tan z$       (iii)  $z \coth z$       (iv)  $\frac{e^z - e}{(1-z)^3}$   
(v)  $\exp(\tan z)$       (vi)  $\sinh \frac{z}{z^2-1}$       (vii)  $\log(1+e^z)$       (viii)  $\tan(z^{-1})$

## Integration and residues

6. Evaluate  $\int z \, dz$  along the straight line from  $-1$  to  $+1$ , and along the semicircular contour in the upper half-plane between the same two points; and evaluate  $\oint_{\gamma} \bar{z} \, dz$  when  $\gamma$  is the circle  $|z| = 1$ , and when  $\gamma$  is the circle  $|z-1| = 1$ .
7. (i) Show that if  $f(z)$  and  $g(z)$  are analytic, and  $g$  has a simple zero at  $z = z_0$ , the residue of  $f(z)/g(z)$  at  $z = z_0$  is  $f(z_0)/g'(z_0)$ . In particular, show that  $f(z)/(z-z_0)$  has residue  $f(z_0)$ .
- (ii) Prove the formula for the residue of a function  $f(z)$  that has a pole of order  $N$  at  $z = z_0$ :

$$\lim_{z \rightarrow z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} ((z-z_0)^N f(z)) \right\}.$$

- (iii) Find the residues of the poles in question 5.

8. Evaluate, using Cauchy's theorem or the residue theorem,

$$(i) \oint_{\gamma_1} \frac{dz}{1+z^2} \quad (ii) \oint_{\gamma_2} \frac{dz}{1+z^2} \quad (iii) \oint_{\gamma_3} \frac{e^z \cot z \, dz}{1+z^2} \quad * \quad (iv) \oint_{\gamma_4} \frac{z^3 e^{1/z} \, dz}{1+z}$$

where  $x = \operatorname{Re}(z)$ ,  $y = \operatorname{Im}(z)$ ,  $\gamma_1$  is the elliptical contour  $x^2 + 4y^2 = 1$ ,  $\gamma_2$  is the circle  $|z| = \sqrt{2}$ ,  $\gamma_3$  is the circle  $|z - (2 + i)| = \sqrt{2}$  and  $\gamma_4$  is the circle  $|z| = 2$ , all traversed anti-clockwise.

9. By integrating the function  $z^n(z-a)^{-1}(z-a^{-1})^{-1}$  around the unit circle and applying the residue theorem, evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a \cos \theta + a^2} \, d\theta$$

where  $a$  is real,  $a > 1$ , and  $n$  is a non-negative integer.

\* Obtain the same result using Cauchy's integral formula instead of the residue theorem.

### The calculus of residues

10. Evaluate  $\int_{-\infty}^{\infty} \frac{dx}{1+x+x^2}$ .

11. By integrating around a keyhole contour, show that

$$\int_0^{\infty} \frac{x^{a-1} \, dx}{1+x} = \frac{\pi}{\sin \pi a} \quad (0 < a < 1).$$

Explain why the given restrictions on the value of  $a$  are necessary.

\* 12. By integrating around a contour involving the real axis and the line  $z = re^{2\pi i/n}$ , evaluate  $\int_0^{\infty} dx/(1+x^n)$ ,  $n \geq 2$ . Check (by change of variable) that your answer agrees with that of the previous question.

13. Establish the following:

$$(i) \int_0^{\infty} \frac{\cos x}{(1+x^2)^3} \, dx = \frac{7\pi}{16e} \quad (ii) \int_0^{\infty} \frac{x^2}{\cosh x} \, dx = \frac{\pi^3}{8}$$

$$(iii) \int_0^{\infty} \frac{\log x}{1+x^2} \, dx = 0 \quad * \quad (iv) \int_0^{\infty} \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2}$$

[For part (ii), use a rectangular contour. For part (iii), integrate  $(\log z)^2/(1+z^2)$  around a keyhole, or  $(\log z)/(1+z^2)$  along the real axis (or both). What goes wrong with  $(\log z)/(1+z^2)$  around a keyhole?]

\* 14. Let  $P(z)$  be a non-constant polynomial. Consider the contour integral  $I = \oint_{\gamma} (P'(z)/P(z)) \, dz$ . Show that, if  $\gamma$  is a contour that encloses no zeros of  $P$ , then  $I = 0$ .

Evaluate the limit of  $I$  as  $R \rightarrow \infty$ , where  $\gamma$  is the circle  $|z| = R$ , and deduce that  $P$  has at least one zero in the complex plane.

15. By considering the integral of  $(\cot z)/(z^2 + \pi^2 a^2)$  around a suitable large contour, prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$$

provided that  $ia$  is not an integer. By considering a similar integral prove also that, if  $a$  is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

Find an expression for  $\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$  and take the limit as  $a \rightarrow 0$  to deduce the value of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .