Complex Methods: Example Sheet 3  
Part IB, Lent Term 2022  
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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Fourier transforms

1. By using the relationship between the Fourier transform and its inverse, show that for real $a$ and $b$ with $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} e^{i\omega t} \, d\omega = \frac{\pi}{a} e^{-a|t|}$$  
and  
$$\int_{-\infty}^{\infty} \frac{b}{(i\omega + a)^2 + b^2} e^{i\omega t} \, d\omega = 2\pi b e^{-at} \sin bt \, H(t)$$

where $H(t)$ is the Heaviside step function. What are the values of the integrals when $a < 0$?  
What happens when $a = 0$?

2. Show that the convolution of the function $e^{-|x|}$ with itself is given by $f(x) = (1 + |x|)e^{-|x|}$. Use the convolution theorem for Fourier transforms to show that

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(1 + k^2)^{\frac{3}{2}}} \, dk$$

and verify this result by contour integration.

3. Let

$$f(x) = \begin{cases} 1 & |x| < \frac{1}{2}a, \\ 0 & \text{otherwise;} \end{cases} \quad g(x) = \begin{cases} a - |x| & |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\tilde{f}(k) = \frac{2}{k} \sin \frac{ak}{2} \quad \text{and} \quad \tilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}.$$  

What is the convolution of $f$ with itself? Use Parseval’s identity to evaluate $\int_{-\infty}^{\infty} (\sin^2 x)/x^2 \, dx$.  
Verify by contour integration the inversion formula for $f(x)$ for all values of $x$ except $\pm \frac{1}{2}a$.

* Verify the inversion formula also at $x = \pm \frac{1}{2}a$.

* 4. The displacement $x(t)$ of a damped harmonic oscillator obeys the equation

$$\dddot{x} + 2\gamma \ddot{x} + q^2 x = f(t), \quad \text{where } \gamma > 0.$$  

Assuming that the Fourier transforms $\tilde{x}(\omega)$ and $\tilde{f}(\omega)$ exist, show that

$$x(t) = \int_{-\infty}^{\infty} G(t - t') f(t') \, dt', \quad \text{where} \quad G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{q^2 + 2i\gamma \omega - \omega^2} \, d\omega.$$  

Show, by differentiation under the integral sign, that

$$\frac{d^2}{dt^2} G(t) + 2\gamma \frac{d}{dt} G(t) + q^2 G(t) = \delta(t).$$

Show that for $0 < \gamma < q$,

$$G(t) = \frac{1}{p} e^{-\gamma t} \sin(pt) \, H(t), \quad \text{where} \quad p = \sqrt{q^2 - \gamma^2}.$$  

[You may use here the results from question 1.]
Laplace transforms

5. Starting from the Laplace transform of 1 (namely \( s^{-1} \)), and using only standard properties of the Laplace transform (shifting, etc.), find the Laplace transforms of the following functions:
   (i) \( e^{-2t} \); (ii) \( t^3 e^{-3t} \); (iii) \( e^{3t} \sin 4t \); (iv) \( e^{-4t} \cosh 4t \); (v) \( e^{-t}H(t-1) \), where \( H \) is the Heaviside step function.

6. Using partial fractions and expressions for the Laplace transforms of elementary functions, find the inverse Laplace transform of \( F(s) = (s + 3)/(s - 2)(s^2 + 1) \). Verify this result using the Bromwich inversion formula.

7. Use Laplace transforms to solve the differential equation
   \[
   \frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y = t^2 e^t
   \]
   with initial conditions \( y(0) = 1, \dot{y}(0) = 0, \ddot{y}(0) = -2 \).

8. Consider a linear system obeying the differential equation
   \[
   \ddot{y} - 3\dot{y} + 2y = u(t), \quad \dot{y}(0) = y(0) = 0.
   \]
   Use Laplace transforms to determine the response of the system to the signal \( u(t) = t \). Determine also the response \( y(t) \) to a signal \( u(t) = \delta(t) \).

   [For \( \delta(t) \), take the Laplace transform to be \( F(s) = \int_0^\infty f(t)e^{-st}dt \), i.e. start “just left of 0”.

9. Solve the integral equation \( f(t) + 4 \int_0^t (t - \tau)f(\tau)\,d\tau = t \) for the unknown function \( f \). Verify your solution.

* 10. The zeroth order Bessel function \( J_0(x) \) satisfies the differential equation
    \[
    x J''_0 + J'_0 + x J_0 = 0
    \]
    for \( x > 0 \), with \( J_0(0) = 1 \) (and \( J'_0(0) = 0 \) from the equation). Find the Laplace transform of \( J_0 \) and deduce that \( \int_0^\infty J_0(x)\,dx = 1 \). Find the convolution of \( J_0 \) with itself.

11. Use Laplace transforms to solve the heat equation \( \partial T/\partial t = \partial^2 T/\partial x^2 \) with boundary conditions
    \( T(x, 0) = \sin^2\pi x \) (\( 0 < x < 1 \)), \( T(0, t) = T(1, t) = 0 \) (\( t > 0 \)). [Hint: \( \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \)]

12. Using the equality \( \int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi} \), find the Laplace transform of \( f(t) = t^{-\frac{1}{2}} \). By integrating around a Bromwich keyhole contour, verify the inversion formula for \( f(t) \). What is the Laplace transform of \( t^{\frac{1}{2}} \)?

* 13. The gamma and beta functions are defined for \( z, w \in \mathbb{C} \) by
    \[
    \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt \quad \text{and} \quad B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt
    \]
    when \( \text{Re}(z), \text{Re}(w) > 0 \). Show that \( \Gamma(z + 1) = z\Gamma(z) \) and hence that \( \Gamma(n + 1) = n! \) if \( n \) is a non-negative integer. Using the previous question, write down the value of \( \Gamma(\frac{1}{2}) \).

   For a fixed value of \( z \), find the Laplace transform of \( f(t) = t^{z-1} \) in terms of \( \Gamma(z) \). Find the Laplace transform of the convolution \( t^{z-1} * t^{w-1} \). Hence establish that
   \[
   B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.
   \]

   The domain of \( \Gamma \) and \( B \) can be extended to the whole of \( \mathbb{C} \), apart from isolated singularities, by analytic continuation. Does the relation (*) still hold?