Complex Methods: Example Sheet 3  
Part IB, Lent Term 2021  
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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Fourier transforms

1. By using the relationship between the Fourier transform and its inverse, show that for real \( a \) and \( b \) with \( a > 0 \),

\[
\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} e^{i\omega t} \, d\omega = \frac{\pi e^{-a|t|}}{a} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{b}{(i\omega + a)^2 + b^2} e^{i\omega t} \, d\omega = 2\pi e^{-at} \sin bt \, H(t)
\]

where \( H(t) \) is the Heaviside step function. What are the values of the integrals when \( a < 0 \)? What happens when \( a = 0 \)?

2. Show that the convolution of the function \( e^{-|x|} \) with itself is given by \( f(x) = (1 + |x|)e^{-|x|} \). Use the convolution theorem for Fourier transforms to show that

\[
f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + k^2} \, dk
\]

and verify this result by contour integration.

3. Let

\[
f(x) = \begin{cases} 1 & |x| < \frac{1}{2}a, \\ 0 & \text{otherwise;}
\end{cases} \quad g(x) = \begin{cases} a - |x| & |x| < a, \\ 0 & \text{otherwise.}
\end{cases}
\]

Show that

\[
\tilde{f}(k) = \frac{2}{k} \sin \frac{ak}{2} \quad \text{and} \quad \tilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}.
\]

What is the convolution of \( f \) with itself? Use Parseval’s identity to evaluate \( \int_{-\infty}^{\infty} (\sin^2 x) / x^2 \, dx \). Verify by contour integration the inversion formula for \( f(x) \) for all values of \( x \) except \( \pm \frac{1}{2}a \).

∗ Verify the inversion formula also at \( x = \pm \frac{1}{2}a \).

4. The displacement \( x(t) \) of a damped harmonic oscillator obeys the equation

\[
\ddot{x} + 2\gamma \dot{x} + q^2 x = f(t), \quad \text{where} \quad \gamma > 0.
\]

Assuming that the Fourier transforms \( \tilde{x}(\omega) \) and \( \tilde{f}(\omega) \) exist, show that

\[
x(t) = \int_{-\infty}^{\infty} G(t - t') f(t') \, dt', \quad \text{where} \quad G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{q^2 + 2i\gamma \omega - \omega^2} \, d\omega.
\]

Show, by differentiation under the integral sign, that

\[
\frac{d^2}{dt^2} G(t) + 2\gamma \frac{d}{dt} G(t) + q^2 G(t) = \delta(t).
\]

Show that for \( 0 < \gamma < q \),

\[
G(t) = \frac{1}{p} e^{-\gamma t} \sin(pt) \, H(t), \quad \text{where} \quad p = \sqrt{q^2 - \gamma^2}.
\]

[You may use here the results from question 1.]
Laplace transforms

5. Starting from the Laplace transform of 1 (namely \(s^{-1}\)), and using only standard properties of the Laplace transform (shifting, etc.), find the Laplace transforms of the following functions:
   (i) \(e^{-2t}\); (ii) \(t^3e^{-3t}\); (iii) \(e^{3t}\sin 4t\); (iv) \(e^{-4t}\cosh 4t\); (v) \(e^{-t}H(t-1)\), where \(H\) is the Heaviside step function.

6. Using partial fractions and expressions for the Laplace transforms of elementary functions, find the inverse Laplace transform of \(F(s) = (s+3)/((s-2)(s^2+1))\). Verify this result using the Bromwich inversion formula.

7. Use Laplace transforms to solve the differential equation
   \[
   \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2 e^t
   \]
   with initial conditions \(y(0) = 1, \dot{y}(0) = 0, \ddot{y}(0) = -2\).

8. Consider a linear system obeying the differential equation
   \[
   \dddot{y} - 3\ddot{y} + 2\dot{y} = u(t), \quad \dot{y}(0) = y(0) = 0.
   \]
   Use Laplace transforms to determine the response of the system to the signal \(u(t) = t\). Determine also the response \(y(t)\) to a signal \(u(t) = \delta(t)\).
   \[\text{[For} \delta(t), \text{take the Laplace transform to be} \ F(s) = \int_0^\infty f(t) e^{-st} \, dt, \text{i.e. start “just left of 0”]}\]

9. Solve the integral equation \(f(t) + 4 \int_0^t (t - \tau) f(\tau) \, d\tau = t\) for the unknown function \(f\). Verify your solution.

* 10. The zeroth order Bessel function \(J_0(x)\) satisfies the differential equation
   \[
xJ_0'' + J_0' + xJ_0 = 0
   \]
   for \(x > 0\), with \(J_0(0) = 1\) (and \(J_0'(0) = 0\) from the equation). Find the Laplace transform of \(J_0\) and deduce that \(\int_0^\infty J_0(x) \, dx = 1\). Find the convolution of \(J_0\) with itself.

11. Use Laplace transforms to solve the heat equation \(\partial T/\partial t = \partial^2 T/\partial x^2\) with boundary conditions \(T(x, 0) = \sin^3 \pi x \quad (0 < x < 1), T(0, t) = T(1, t) = 0 \quad (t > 0)\). \([\text{Hint:} \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]\]

12. Using the equality \(\int_0^\infty e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi}\), find the Laplace transform of \(f(t) = t^{-\frac{1}{2}}\). By integrating around a Bromwich keyhole contour, verify the inversion formula for \(f(t)\). What is the Laplace transform of \(t^{\frac{1}{2}}\)?

* 13. The gamma and beta functions are defined for \(z, w \in \mathbb{C}\) by
   \[
   \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad \text{and} \quad B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} \, dt
   \]
   when \(\Re(z), \Re(w) > 0\). Show that \(\Gamma(z + 1) = z\Gamma(z)\) and hence that \(\Gamma(n + 1) = n!\) if \(n\) is a non-negative integer. Using the previous question, write down the value of \(\Gamma\left(\frac{1}{2}\right)\).
   For a fixed value of \(z\), find the Laplace transform of \(f(t) = t^{z-1}\) in terms of \(\Gamma(z)\). Find the Laplace transform of the convolution \(t^{z-1} \ast t^{w-1}\). Hence establish that
   \[
   B(z, w) = \frac{\Gamma(z) \Gamma(w)}{\Gamma(z + w)}.
   \]
   The domain of \(\Gamma\) and \(B\) can be extended to the whole of \(\mathbb{C}\), apart from isolated singularities, by analytic continuation. Does the relation (*) still hold?