# COMPLEX METHODS 3

**Fourier Transforms**
In all cases, the Fourier transform is denoted by a tilde and defined by $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$.

**1** If

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} a - |x| & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a \end{cases}$$

show that $\tilde{f}(k) = \dfrac{2\sin ak}{k}$ and $\tilde{g}(k) = \dfrac{2}{k^2}\,(1 - \cos(ka))$.

Verify (by contour integration) that the inversion formula for $f(x)$ holds **for all** real $k$.

**2** Using the Fourier inversion formula, show that, for $a > 0$,

(i) $e^{-a|t|} = \dfrac{a}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2}\, e^{i\omega t}\mathrm{d}\omega$ , (ii) $H(t)e^{-at}\sin bt = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{b}{(i\omega + a)^2 + b^2}\, e^{i\omega t}\mathrm{d}\omega$ ,

where $H(t)$ is the Heaviside step function. What goes wrong with the integrals if $a = 0$? What are the values of the integrals when $a < 0$?

**3** The displacement $x(t)$ of a damped harmonic oscillator obeys the equation

$$\ddot{x} + 2\gamma\dot{x} + q^2 x = f(t) \qquad (\gamma > 0) \ ,$$

and $\tilde{x}(\omega)$ and $\tilde{f}(\omega)$ exist. Show that

$$x(t) = \int_{-\infty}^{\infty} G(t - t')f(t')\mathrm{d}t' \ , \qquad \text{where} \qquad G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{q^2 + 2i\gamma\omega - \omega^2}e^{i\omega t}\mathrm{d}\omega \ .$$

Verify formally, by differentiation under the integral sign, that

$$\frac{d^2}{dt^2}G(t) + 2\gamma\frac{d}{dt}G(t) + q^2 G(t) = \delta(t) \ .$$

Assume that the system is underdamped, that is that $0 < \gamma < q$, and let $p = \sqrt{q^2 - \gamma^2}$ . Deduce (see question 2) that

$$G(t) = \frac{1}{p}\, e^{-\gamma t}\sin pt\, H(t) \ .$$

**4** Show that the convolution, $h(x)$, of the function $f(x) = e^{-|x|}$ with itself is given by

$$h(x) = \begin{cases} (1-x)e^{x} & \text{for } x < 0 \\ (1+x)e^{-x} & \text{for } x > 0. \end{cases}$$

Use the convolution theorem to show that

$$h(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(1+k^2)^2} \mathrm{d}k \, .$$

Verify this result by contour integration.

**5** Let $f(x) = g(x)e^{-a|x|}$, where $g(x + 2\pi) = g(x)$ and $a > 0$. Show that the Fourier transform of $f(x)$ is given by

$$\tilde{f}(k) = \frac{c(k-ia)}{1 - e^{-2\pi i(k-ia)}} - \frac{c(k+ia)}{1 - e^{-2\pi i(k+ia)}} \, ,$$

where the function $c$ is defined by $c(k) = \int_0^{2\pi} g(x)e^{-ikx}\mathrm{d}x$ .

Assuming that $c(k)$ is analytic, give a sketch of the complex $k$-plane, showing the singularities of $\tilde{f}(k)$. Use the Fourier inversion theorem and a contour integral, to show that, for $x > 0$,

$$f(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} c(n)e^{(in-a)x} \, .$$

What is the corresponding result when $x < 0$ ?

Deduce that

$$g(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} c(n)e^{inx} \, .$$

[*This shows how the Fourier-transform representation of a function reduces to a Fourier-series representation if the function is periodic.*]

**Laplace Transforms**

In all cases the Laplace transform of a function $f(t)$ is denoted by $\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^\infty e^{-st} f(t) dt$.

**6** Use the known properties of the Laplace transform to find the Laplace transform of the following functions: (i) $t^3 e^{-3t}$, (ii) $2e^{3t}\sin 4t$, (iii) $e^{-4t}\cosh 2t$.

**7** Use partial fractions and a contour integral to find the inverse Laplace transform of $\frac{s+3}{(s-2)\,(s^2+1)}$.

**8** Using the known result $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$, find the Laplace transforms of $f(t) = t^{-1/2}$ and $f(t) = t^{1/2}$. Verify your results by finding the inverse Laplace transform using a contour integral. (*Hint: use a contour similar to a keyhole in order to avoid the branch cut*).

**9** *Gamma and Beta functions.* The Gamma function is defined by $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$. Show that $\Gamma(n+1) = n\Gamma(n)$ and therefore $\Gamma(n+1) = n!$ if $n$ is a positive integer. Show that $\Gamma(-1/2) = -2\sqrt{\pi}$, $\Gamma(1/2) = \sqrt{\pi}$ and that $\Gamma(n)$ is singular for negative integer values of $n$. The related beta function is defined as $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$. Use the convolution theorem for Laplace transforms to establish that

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

**10** Find the Laplace transform of the zeroth order Bessel function $J_0(t)$ by using its series expansion:

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots$$

and also by using the differential equation satisfied by $J_0(t)$, $tJ_0'' + J_0' + tJ_0 = 0$. Use this to show that $\int_0^\infty J_0(t)dt = 1$. Compute the convolution of $J_0$ with itself.

**11** Use Laplace transforms to solve the differential equation:

$$y''' - 3y'' + 3y' - y \;=\; t^2 e^t,$$

with initial conditions $y(0) = 1, y'(0) = 0, y''(0) = -2$.

**12** Use Laplace transforms to solve the following differential equation:

$$ty'' + y' + 4ty = 0,$$

with initial conditions $y(0) = 3, y'(0) = 0$.

**13** Use Laplace transforms to solve the heat equation $\partial T/\partial t = \partial^2 T/\partial x^2$, with the boundary conditions: $T(x,0) = 3\sin 2\pi x$, $T(0,t) = T(1,t) = 0$ for $0 < x < 1$, $t > 0$.

**14** A linear system is described by the differential equation

$$y'' - 3y' + 2y \;=\; u(t),$$

with initial conditions $y'(0) = y(0) = 0$. Use Laplace transforms to determine the response of the system to the signal $u(t) = t$. Determine also the impulse response (the response $y(t)$ to a signal $u(t) = \delta(t)$).

**15** Solve $f(t) + 4\int_0^t (t-\tau)f(\tau)d\tau = t$. Verify your solution.

**16** Solve

$$x' = 10y - 5x; y' = y - x$$

With $x(0) = 3, y(0) = 1$.

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