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1. Recall from lectures $c_n(\theta) = \cos(2\pi n \theta / L)$ and $s_n(\theta) = \sin(2\pi n \theta / L)$. Verify the orthogonality relations
\[ \langle c_n, c_m \rangle = \langle s_n, s_m \rangle = \frac{1}{L} \delta_{mn}, \quad \langle c_n, s_m \rangle = \delta_{n,m} - \frac{1}{L} \delta_{mn}, \quad \langle c_n, s_m \rangle = 0 \quad m, n \geq 1 \]
where \( \langle f, g \rangle = \int_0^L f(\theta)g(\theta)d\theta \). This confirms \( \{1, c_n, s_n\}_{n=1}^\infty \) are orthogonal, as stated in lectures.

2. Consider 2-periodic function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(\theta) = (1 - \theta^2)^2 \) when \( \theta \in [-1, 1] \). Show that it has Fourier series
\[ f(\theta) \sim \frac{8}{15} + \frac{48}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos(n\pi \theta). \]
Can we replace ‘~’ with ‘=’ in this case? Sketch the graph of \( f \) and comment on the number of continuous derivatives it has and the relation to the decay of the Fourier coefficients.

3. Suppose \( f(\theta) = \theta^2 \) when \( \theta \in [0, \pi] \).
   (i) Construct (a) the Sine series for \( f \) and (b) the Cosine series for \( f \), each having period 2\( \pi \). Sketch the 2\( \pi \)-periodic functions obtained in (a) and (b) in the range \( \theta \in [-\pi, \pi] \).
   (ii) If the series in (a) and (b) are formally differentiated term-by-term, are the resulting series related to the Fourier series for 2\( \pi \)-periodic functions \( g, h : \mathbb{R} \to \mathbb{R} \) for which \( g(\theta) = 2\theta \) and \( h(\theta) = 2\theta |\theta| \) when \( \theta \in [-\pi, \pi] \)?

4. Find the complex Fourier series for the 2\( \pi \)-periodic function \( f : \mathbb{R} \to \mathbb{R} \) for which \( f(\theta) = e^{i\theta} \) when \( \theta \in [-\pi, \pi] \). Using Parseval’s theorem, deduce that
\[ \sum_{n=1}^{\infty} \frac{1}{1 + n^2} = \frac{1}{2} \left( \pi \coth \pi - 1 \right). \]
Obtain the same result by evaluating the complex Fourier series at an appropriate point in \([-\pi, \pi]\).

5. We say a sequence \( \{r_n\}_{n \in \mathbb{Z}} \) decays rapidly if \( |n|^k r_n \to 0 \) as \( |n| \to \infty \) for every \( k \geq 0 \).
   (i) Let \( f \) be a smooth, 2\( \pi \)-periodic function. Show that the complex Fourier coefficients \( \{\hat{f}_n\} \) decay rapidly.
   (ii) Construct an \( \mathbb{L} \)-periodic function with rapidly decaying, non-zero complex Fourier coefficients.

6. By considering the Sturm-Liouville problem for \( y = y(x) \)
\[ \begin{cases} y'' + \lambda y = 0, & 0 < x < L, \\ y'(0) = 0, \\ y'(L) = 0, \end{cases} \]
re-derive the Cosine series representation for any \( f \in C^2[0, L] \) with \( f'(0) = f'(L) = 0 \).

7. Prove that the boundary value problem for \( y = y(x) \)
\[ \begin{cases} y'' + \lambda y = 0, & 0 < x < 1, \\ y(0) = 0, \\ y(1) + y'(1) = 0, \end{cases} \]
has infinitely many eigenvalues \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) and indicate roughly the behaviour of \( \lambda_n \) as \( n \to \infty \).

8. Express the following eigenvalue problems as Sturm-Liouville problems on \([-1, 1]\) and \([0, 1]\), respectively:
   (i) \( (1 - x^2) y'' - 2xy' + \lambda y = 0 \),  \( x(1-x)y'' - (ax-b)y' + \lambda y = 0 \),
where \( a > b > 0 \) are constant and \( \lambda \) is constant. Are either of these problems singular?
   (iii) Find the eigenvalues and eigenfunctions of the boundary value problem for \( y = y(x) \)
\[ \begin{cases} y'' + (4+\lambda)y = 0, & 0 < x < 1, \\ y(0) = 0, \\ y(1) = 0. \end{cases} \]
What is the orthogonality relation for these eigenfunctions?
9. Define the functions $q_n(x) = \frac{1}{2 \pi} \left( \frac{d}{dx} \right)^n \left( x^2 - 1 \right)^n$ for $n = 1, 2, \ldots$

(a) Show that $q_n(x)$ is a polynomial of degree $n$;
(b) Deduce that $q_n = P_n$;
(c) $q_n(1) = 1$ for all $n$;
(d) $q_n$ satisfies Legendre’s equation.

Hint: for (a)(iii) show $u_n = (x^2 - 1)^n$ satisfies $(x^2 - 1)u_n' - 2xu_n = 0$ and differentiate further.

10. Recall from lectures that if $u_n(x) = J_m(xr)$, where $J_m$ is the $m$th order Bessel function of the 1st kind, then

$$\frac{d}{dx} \left( r \frac{dy_n}{dr} \right) + \frac{m^2}{r} y_n = \alpha^2 r y_n, \quad r \in (0, 1).$$

Show that if $y_\beta(x) = J_m(\beta x)$ then $[r(y_\alpha y_\beta' - y_\beta y_\alpha')]' = (\alpha^2 - \beta^2) r y_\alpha y_\beta$. Deduce that

$$\int_0^1 J_m(\alpha x) J_m(\beta x) r \, dx = \frac{\beta J_m(\alpha) J_m'(\beta) - \alpha J_m(\beta) J_m'(\alpha)}{\alpha^2 - \beta^2}, \quad \alpha \neq \beta.$$

Use this result to show that $\int_0^1 J_m(j_{mk} x) J_m(j_{mk} x) r \, dx = \frac{1}{2} [J_{m}(j_{mk})]^2 \delta_{kl}$, where $J_m(j_{mk}) = 0$, $k = 1, 2, \ldots$

Additional problems

These questions should not be attempted at the expense of earlier ones.

11. Let $f$ be the $2\pi$-periodic square wave for which $f(\theta) = 1$ on $[0, \pi)$ and $f(\theta) = 0$ on $[\pi, 2\pi)$.

(i) Sketch the graph of $f$ and show that

$$f(\theta) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}.$$

(ii) Let $S_N f$ denote the partial Fourier series for $f$. By considering $\sum_{n=1}^{N} \cos[(2n-1)\theta]$, or otherwise, show

$$(S_N f)(\theta) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\theta} \frac{\sin(2N\phi)}{\sin \phi} \, d\phi.$$

(iii) Deduce that $(S_N f)(\theta)$ has a local extrema at $\theta = 2\pi m / 2N$, $m \in \mathbb{Z} \setminus 2N\mathbb{Z}$ and that for large $N$

$$(S_N f) \left( \frac{\pi}{2N} \right) \approx \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\pi/2} \frac{\sin u}{u} \, du = \frac{1}{2} + \int_{0}^{\pi/2} \frac{\sin u}{u(\pi - u)} \, du \geq 1.08.$$

Hint for lower bound: $\sin u \geq u - u^3/3!$. Comment on the accuracy of partial Fourier series at discontinuities.

12. Set $V = \{ y \in C^2[a, b] : y(a) = y(b) = 0 \}$ (i.e. Dirichlet boundary conditions) and let $L = \frac{1}{w} \left[ -\frac{d}{dx} \left( p \frac{dy}{dx} \right) + q \right]$ be a Sturm-Liouville operator with $p, q, w$ smooth and $p, w > 0$ on $[a, b]$. Consider the Rayleigh quotient

$$R[y] = \frac{\int_{a}^{b} \left( p (y')^2 + q y^2 \right) \, dx}{\int_{a}^{b} w y^2 \, dx}, \quad y \in V.$$

(a) By considering $\langle Ly, y \rangle_{w^*}$ show that if $y \in V$ satisfies $Ly = \lambda y$ then $\lambda = R[y]$.
(b) Let $\lambda_1 = \inf_{y \in V \setminus \{0\}} R[y]$ and suppose that there exists a $y_1 \in V$ such that $R[y_1] = \lambda_1$. If we set

$$F(\epsilon) = R[y_1 + \epsilon \eta],$$

where $\eta \in V$, explain why $F'(0) = 0$. Hence show that $L y_1 = \lambda_1 y_1$. Comment on this result in relation to finding the smallest eigenvalue of $L$. How might you try to find the second smallest? (Hint: orthogonality).
(c) Take $[a, b] = [0, 1]$ and $L = -d^2/dx^2$. Compute $R[y]$ where $y(x) = x(1-x)$ and deduce $\lambda_{\text{min}} = \pi^2 \leq 10$.

You may assume that if $\int_{a}^{b} f \eta \, dx = 0$ for all $\eta \in V$ then $f = 0$. 