METHODS — EXAMPLES I

Fourier series

1. *Fourier coefficients (full-range series).* For the periodic function $f(x) = (x^2 - 1)^2$ on the interval $-1 \le x < 1$, show that it has the Fourier series

$$f(x) = \frac{8}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos n\pi x\,.$$

Sketch the function $f(x)$ and comment on its differentiability and the order of the terms in its Fourier series as $n \to \infty$.

2. *Fourier coefficients (half-range series).* Suppose that $f(x) = x^2$ for $0 \le x \le \pi$. Express $f(x)$ as (a) a Fourier sine series, and (b) a cosine series, each having period 2π. Sketch the functions represented by (a) and (b) in the range -6π to 6π. If the series (a) and (b) are differentiated term-by-term, how are the answers related (if at all) to the Fourier series for $g(x) = 2x$ and $h(x) = 2|x|$ each in the range $(-\pi, \pi)$?

3. *Series summation.* Find the Fourier series of $f(x) = e^x$ on $(-\pi, \pi)$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1) \quad .$$

4. *Parseval's identity and a low pass filter.* (i) Given that a function $f(t)$ defined over the interval $(-T, T)$ has the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi t}{T}) + b_n \sin(\frac{n\pi t}{T}) \right], \qquad \text{show that} \qquad \frac{1}{T}\int_{-T}^{T} [f(t)]^2 dt = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\,,$$

where you may assume $f(t)$ is such that this series is convergent.
(ii) A unit amplitude square wave of period $2T$ is given by $f(t) = 1$ for $0 < t < T$ and $f(t) = -1$, for $-T < t < 0$. Suppose this is the input for a system which permits angular frequencies less than $\frac{9}{2}\pi T^{-1}$ to be perfectly transmitted and frequencies greater than $\frac{9}{2}\pi T^{-1}$ to be perfectly absorbed. Calculate the form of the output. The power is proportional to the mean value of $f^2(t)$; what fraction of the incident power is transmitted?

5. *Discontinuities and the Wilbraham-Gibbs phenomenon*.* (i) Suppose that f is a square wave given by

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi\,. \end{cases} \qquad \text{Sketch } f \text{ and show that} \quad f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}\,.$$

(ii) Now define the partial sum of this series as

$$S_N(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{N} \frac{\sin(2n-1)x}{2n-1}\,,$$

and find the following expression

$$S_N(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^x \frac{\sin 2Nt}{\sin t} dt \quad .$$

[Hint: consider $\sum_{n=1}^{N} \cos(2n-1)x$.]

(iii) Deduce that $S_N(x)$ has extrema at $x = m\pi/2N$, $m = 1, 2, ...2N-1, 2N+1, ...$, (i.e. all integer m except $m = 2kN$, k integer) and that the height of the first maximum for large N is approximately

$$S_N(\frac{\pi}{2N}) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin u du}{u} (\simeq 1{\cdot}089)\,.$$

Comment on the accuracy of Fourier series at discontinuities. (This question takes you through some important steps which are used in the proof of Fourier's theorem - refer, for example, to chapter 14 of Jeffreys & Jeffreys.)

Sturm-Liouville theory

6. *Eigenfunctions and eigenvalues*. Prove that the boundary value problem

$$y'' + \lambda y = 0; \; y(0) = 0, \quad y(1) + y'(1) = 0,$$

has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$... Indicate roughly the behaviour of λ_n as $n \to \infty$.

7. *Recasting in Sturm-Liouville form*. Express the following equations in Sturm-Liouville form:

(i) $(1-x^2)y'' - 2xy' + n(n+1)y = 0\,,$ (ii) $x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0\,,$

where $n, a, b,$ and c are constants.

(iii) Find the eigenvalues and eigenfunctions for

$$y'' + 4y' + (4+\lambda)y = 0, \; y(0) = y(1) = 0.$$

What is the orthogonality relation for these eigenfunctions?

8. *Bessel's equation*. (i) Show that the eigenvalues of the Sturm-Liouville problem

$$\frac{d}{dx}(x\frac{du}{dx}) + \lambda x u = 0\,, \qquad 0 < x < 1\,,$$

with $u(x)$ bounded as $x \to 0$ and $u(1) = 0$, are $\lambda = j_n^2$ $(n = 1, 2, ...)$, where the j_n are the zeros of the Bessel function $J_0(z)$, arranged in ascending order. [Note: Bessel's equation of order zero is $\frac{d}{dz}(z\frac{dy}{dz}) + zy = 0\,, (z > 0)$, which you may assume has one solution $J_0(z)$ defined as

$$J_0(z) = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{2^{2m}(m!)^2}$$

and a second solution that is the sum of a regular function and $J_0(z)\log z$.]

(ii) Using integration by parts on the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$, show that

$$\int_0^1 J_0(\alpha x)J_0(\beta x)x dx = \frac{\beta J_0(\alpha)J_0'(\beta) - \alpha J_0(\beta)J_0'(\alpha)}{\alpha^2 - \beta^2} \quad (\beta \neq \alpha)$$

$$\int_0^1 J_0(j_n x)J_0(j_m x)x dx = 0\,, \quad (n \neq m)\,, \qquad \int_0^1 [J_0(j_n x)]^2 x dx = \frac{1}{2}[J_0'(j_n)]^2.$$ [*Hint:* Consider $\beta = j_n + \epsilon$ as $\epsilon \to 0$.]

(iii) Assume that the inhomogeneous equation

$$\frac{d}{dx}(x\frac{du}{dx}) + \tilde{\lambda} x u = x f(x),$$

where $\tilde{\lambda}$ is not an eigenvalue, has a unique solution such that $u(x)$ is bounded as $x \to 0$ and $u(1) = 0$. Assuming also that $f(x)$ satisfies the same boundary conditions as u and the completeness of the eigenfunctions $J_0(j_n x)$, obtain the eigenfunction expansion of u.

9. *Higher order self-adjoint form**. Consider the fourth-order differential operator

$$\mathcal{L} = \sum_{r=0}^{4} p_r(x)\frac{d^r}{dx^r},$$

where the $p_r(x)$ are real functions, with BCs $y(0) = y(1) = y'(0) = y'(1) = 0$. Show that $\mathcal{L}$ is self-adjoint if and only if $p_3 = 2p_4'$, $p_1 = p_2' - p_4'''$.

Considering a specific example, show that the boundary value problem

$$-y'''' + \lambda y = 0; \; y(0) = y(1) = y'(0) = y'(1) = 0$$

has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$... . Indicate roughly the behaviour of λ_n as $n \to \infty$.