METHODS — EXAMPLES I

Fourier series

1. **Fourier coefficients (full-range series).** For the periodic function \( f(x) = (x^2 - 1)^2 \) on the interval \(-1 \leq x < 1\), show that it has the Fourier series

\[
f(x) = \frac{8}{15} + \frac{48}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos n\pi x.
\]

Sketch the function \( f(x) \) and comment on its differentiability and the order of the terms in its Fourier series as \( n \to \infty \).

2. **Fourier coefficients (half-range series).** Suppose that \( f(x) = x^2 \) for \( 0 \leq x < \pi \). Express \( f(x) \) as (a) a Fourier sine series, and (b) a cosine series, each having period \( 2\pi \). Sketch the functions represented by (a) and (b) in the range \(-6\pi \) to \( 6\pi \). If the series (a) and (b) are differentiated term-by-term, how are the answers related (if at all) to the Fourier series for \( g(x) = 2x \) and \( h(x) = 2|x| \) each in the range \([-\pi, \pi]\)?

3. **Series summation.** Find the Fourier series of the \( 2\pi \)-periodic function \( f(x) = e^x \) on \([-\pi, \pi]\). Deduce that

\[
\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} (\pi \coth \pi - 1).
\]

4. **Parseval’s identity and a low pass filter.** (i) Given that a function \( f(t) \) defined over the interval \((-T, T)\) has the Fourier series

\[
f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi t}{T} \right) + b_n \sin \left( \frac{n\pi t}{T} \right) \right],
\]

show that

\[
\frac{1}{T} \int_{-T}^{T} |f(t)|^2 dt = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),
\]

where you may assume \( f(t) \) is such that this series is convergent.

(ii) A unit amplitude square wave of period \( 2T \) is given by \( f(t) = 1 \) for \( 0 < t < T \) and \( f(t) = -1 \) for \(-T < t < 0\). Suppose this is the input for a system which permits angular frequencies less than \( \frac{\pi}{2} T^{-1} \) to be perfectly transmitted and frequencies greater than \( \frac{\pi}{2} T^{-1} \) to be perfectly absorbed. Calculate the form of the output. The power is proportional to the mean value of \( f^2(t) \); what fraction of the incident power is transmitted?

5. **Discontinuities and the Wilbraham-Gibbs phenomenon.** (i) Suppose that \( f \) is a \( 2\pi \)-periodic square wave given by

\[
f(x) = \begin{cases} 
1 & 0 < x < \pi \\
0 & \pi < x < 2\pi.
\end{cases}
\]

Sketch \( f \) and show that its Fourier series is

\[
\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.
\]

(ii) Now define the partial sum of this series as

\[
S_N(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin(2n-1)x}{2n-1},
\]

and find the following expression

\[
S_N(x) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{x} \frac{\sin 2Nt}{\sin t} dt.
\]

[Hint: consider \( \sum_{n=1}^{N} \cos(2n-1)x \).]

(iii) Deducate that \( S_N(x) \) has extrema at \( x = m\pi/2N, \ m = 1, 2, \ldots 2N - 1, 2N + 1, \ldots \) (i.e. all integer \( m \) except \( m = 2kN, \ k \) integer) and that the height of the first maximum for large \( N \) is approximately

\[
S_N\left(\frac{\pi}{2N}\right) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\pi/2N} \frac{\sin u}{u} du \ (\approx 1.089).
\]

Comment on the accuracy of Fourier series at discontinuities.

[This question takes you through some important steps which are used in the proof of Fourier’s theorem - refer, for example, to chapter 14 of Jeffreys & Jeffreys. Henry Wilbraham was a fellow of Trinity who wrote a paper (when he was 22) deriving this result 50 years before Gibbs, after whom it is commonly named.]
Sturm-Liouville theory

6. **Eigenfunctions and eigenvalues.** Prove that the boundary value problem

\[ y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) + y'(1) = 0, \]

has infinitely many eigenvalues \( \lambda_1 < \lambda_2 < \lambda_3 \ldots. \) Indicate roughly the behaviour of \( \lambda_n \) as \( n \to \infty. \)

7. **Recasting in Sturm-Liouville form.** Express the following equations in Sturm-Liouville form:

(i) \((1 - x^2)y'' - 2xy' + \lambda y = 0\), \hspace{1cm} (ii) \(x(x-1)y'' + [ax-b]y' + \lambda y = 0\),

where \( a > b \) and \( \lambda \) are constants.

(iii) Find the eigenvalues and eigenfunctions for

\[ y'' + 4y' + (4 + \lambda)y = 0, \quad y(0) = y(1) = 0. \]

What is the orthogonality relation for these eigenfunctions?

8. **Bessel’s equation.** (i) Show that the eigenvalues of the Sturm-Liouville problem

\[ \frac{d}{dx} \left( x \frac{du}{dx} \right) + \lambda xu = 0, \quad 0 < x < 1, \]

with \( u(x) \) bounded as \( x \to 0 \) and \( u(1) = 0 \), are \( \lambda = j_n^2 \) \((n = 1, 2, \ldots)\), where the \( j_n \) are the zeros of the Bessel function \( J_0(z) \), arranged in ascending order. [Note: Bessel’s equation of order zero is \( \frac{d}{dz}(z \frac{dy}{dz}) + y = 0, (z > 0) \), which you may assume has one solution \( J_0(z) \) defined as

\[ J_0(z) = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{2^{2m} m!^2}, \]

and a second solution that is the sum of a regular function and \( J_0(z) \log z \).]

(ii) Using integration by parts on the differential equations for \( J_0(\alpha x) \) and \( J_0(\beta x) \), show that

\[ \int_0^1 J_0(\alpha x)J_0(\beta x)xdx = \frac{\beta J_0(\alpha)J_1(\beta) - \alpha J_0(\beta)J_1(\alpha)}{\alpha^2 - \beta^2} \quad (\beta \neq \alpha) \]

\[ \int_0^1 J_0(j_n x)J_0(j_m x)xdx = 0, \quad (n \neq m), \quad \int_0^1 [J_0(j_n x)]^2xdx = \frac{1}{2} [J_0'(j_n)]^2. \quad [\text{Hint: Consider } \beta = j_n + \epsilon \text{ as } \epsilon \to 0.] \]

(iii) Assume that the inhomogeneous equation

\[ \frac{d}{dx} \left( x \frac{du}{dx} \right) + \lambda xu = xf(x), \]

where \( \tilde{\lambda} \) is not an eigenvalue, has a unique solution such that \( u(x) \) is bounded as \( x \to 0 \) and \( u(1) = 0 \). Assuming also that \( f(x) \) satisfies the same boundary conditions as \( u \) and the completeness of the eigenfunctions \( J_0(j_n x) \), obtain the eigenfunction expansion of \( u \).

9. **Higher order self-adjoint form.** Show that the fourth-order differential operator

\[ \mathcal{L} = \sum_{r=0}^{4} p_r(x) \frac{d^r}{dx^r}, \]

where the \( p_r(x) \) are real functions, is self-adjoint (for appropriate choices of boundary conditions, which you do not need to determine) only if \( p_3 = 2p_4, \quad p_1 = p_2 - p_4 \).

Considering a specific example, show that the boundary value problem

\[-y''' + \lambda y = 0; \quad y(0) = y(1) = y'(0) = y'(1) = 0,\]

where \( \lambda > 0 \) is real, has infinitely many eigenvalues \( \lambda_1 < \lambda_2 < \lambda_3 \ldots. \) Indicate roughly the behaviour of \( \lambda_n \) as \( n \to \infty. \)

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\(^1\)If you find any errors in the Methods Examples sheets, please inform your supervisor or email epss@damtp.cam.ac.uk.