## Mathematical Tripos Part IB Methods, Example Sheet 1

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**1.** Define  $c_n(\theta) = \cos(2\pi n\theta/L)$  and  $s_n(\theta) = \sin(2\pi n\theta/L)$ . Verify the orthogonality relations

$$\langle c_n, c_m \rangle = \langle s_n, s_m \rangle = \frac{1}{2}L\delta_{mn}, \quad \langle 1, c_n \rangle = \langle 1, s_m \rangle = \langle c_n, s_m \rangle = 0 \quad m, n \ge 1$$

where  $\langle f, g \rangle = \int_0^L f(\theta) \overline{g(\theta)} \, \mathrm{d}\theta$ . This shows  $\{1, c_n, s_n\}_{n=1}^\infty$  are orthogonal.

**2.** Consider 2-periodic function  $f : \mathbf{R} \to \mathbf{R}$  with  $f(\theta) = (1 - \theta^2)^2$  when  $\theta \in [-1, 1)$ . Show that it has Fourier series

$$f(\theta) \sim \frac{8}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos(n\pi\theta).$$

Can we replace ' $\sim$ ' with '=' in this case? Sketch the graph of f and comment on the number of continuous derivatives it has and the relation to the decay of the Fourier coefficients.

**3.** Suppose  $f(\theta) = \theta^2$  when  $\theta \in [0, \pi)$ .

(i) Construct (a) the Sine series for f and (b) the Cosine series for f, each having period  $2\pi$ . Sketch the  $2\pi$ -periodic functions obtained in (a) and (b) in the range  $\theta \in [-6\pi, 6\pi)$ .

(ii) If the series in (a) and (b) are formally differentiated term-by-term, are the resulting series related to the Fourier series for  $2\pi$ -periodic functions  $g, h : \mathbf{R} \to \mathbf{R}$  for which  $g(\theta) = 2\theta$  and  $h(\theta) = 2|\theta|$  when  $\theta \in [-\pi, \pi)$ ?

**4.** Find the complex Fourier series for the  $2\pi$ -periodic function  $f : \mathbf{R} \to \mathbf{R}$  for which  $f(\theta) = e^{\theta}$  when  $\theta \in [-\pi, \pi)$ . Using Parseval's theorem, deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} \left( \pi \coth \pi - 1 \right).$$

Obtain the same result by evaluating the complex Fourier series at an appropriate point in  $[-\pi, \pi)$ .

- 5. We say a sequence  $\{r_n\}_{n \in \mathbb{Z}}$  decays rapidly if  $|n|^k r_n \to 0$  as  $|n| \to \infty$  for every  $k \ge 0$ .
- (i) Let f be a smooth, L-periodic function. Show that the complex Fourier coefficients  $\{\hat{f}_n\}$  decay rapidly.
- (ii) Construct an L-periodic function with rapidly decaying, non-zero complex Fourier coefficients.
- **6.** By considering the Sturm-Liouville problem for y = y(x)

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < L, \\ y'(0) = 0, \\ y'(L) = 0, \end{cases}$$

re-derive the Cosine series representation for any  $f \in C^2[0, L]$  with f'(0) = f'(L) = 0.

7. Prove that the boundary value problem for y = y(x)

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < 1, \\ y(0) = 0, \\ y(1) + y'(1) = 0. \end{cases}$$

has infinitely many eigenvalues  $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$  and indicate roughly the behaviour of  $\lambda_n$  as  $n \to \infty$ .

8. Express the following eigenvalue problems as Sturm-Liouville problems on [-1, 1] and [0, 1], respectively:

(i) 
$$(1-x^2)y'' - 2xy' + \lambda y = 0$$
, (ii)  $x(1-x)y'' - (ax-b)y' + \lambda y = 0$ ,

where a > b > 0 are constant and  $\lambda$  is constant. Are either of these problems singular? (iii) Find the eigenvalues and eigenfunctions of the boundary value problem for y = y(x)

$$\begin{cases} y'' + 4y' + (4+\lambda)y = 0, & 0 < x < 1, \\ y(0) = 0, & \\ y(1) = 0. & \end{cases}$$

What is the orthogonality relation for these eigenfunctions?

**9.** Define the functions  $q_n(x) = \frac{1}{2^n n!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^2 - 1)^n$  for  $n = 1, 2, \ldots$ 

(a) Show (i)  $q_n$  is a polynomial of degree n; (ii)  $q_n(1) = 1$  for all n; (iii)  $q_n$  satisfies Legendre's equation. (b) Deduce (i)  $q_n = P_n$ ; (ii)  $\int_{-1}^1 P_n(x)^2 dx = 2/(2n+1)$ ; (iii)  $\int_{-1}^1 x^m P_n(x) dx = 0$  if m < n.

*Hint*: for (a)(iii) show  $u_n = (x^2 - 1)^n$  satisfies  $(x^2 - 1)u'_n - 2nxu_n = 0$  and differentiate further.

10. Recall from lectures that if  $y_{\alpha}(r) = J_m(\alpha r)$ , where  $J_m$  is the *m*th order Bessel function of the 1st kind, then

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}y_{\alpha}}{\mathrm{d}r}\right) + \frac{m^2}{r}y_{\alpha} = \alpha^2 r y_{\alpha}, \quad r \in (0,1).$$

Show that if  $y_{\beta}(r) = J_m(\beta r)$  then  $[r(y_{\alpha}y'_{\beta} - y_{\beta}y'_{\alpha})]' = (\alpha^2 - \beta^2) ry_{\alpha}y_{\beta}$ . Deduce that

$$\int_0^1 J_m(\alpha r) J_m(\beta r) r \, \mathrm{d}r = \frac{\beta J_m(\alpha) J'_m(\beta) - \alpha J_m(\beta) J'_m(\alpha)}{\alpha^2 - \beta^2}, \quad \alpha \neq \beta$$

Use this result to show that  $\int_0^1 J_m(j_{mk}r)J_m(j_{m\ell}r)r\,dr = \frac{1}{2}[J'_m(j_{mk})]^2\delta_{k\ell}$ , where  $J_m(j_{mk}) = 0, k = 1, 2, ...$ 

## Additional problems

These questions should not be attempted at the expense of earlier ones.

11. Let f be the  $2\pi$ -periodic square wave for which  $f(\theta) = 1$  on  $[0, \pi)$  and  $f(\theta) = 0$  on  $[\pi, 2\pi)$ . (i) Sketch the graph of f and show that

$$f(\theta) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left[(2n-1)\theta\right]}{2n-1}$$

(ii) Let  $S_N f$  denote the partial Fourier series for f. By considering  $\sum_{n=1}^N \cos[(2n-1)\theta]$ , or otherwise, show

$$(S_N f)(\theta) = \frac{1}{2} + \frac{1}{\pi} \int_0^\theta \frac{\sin(2N\phi)}{\sin\phi} \,\mathrm{d}\phi.$$

(iii) Deduce that  $(S_N f)(\theta)$  has a local extrema at  $\theta = \pi m/2N, m \in \mathbb{Z} \setminus 2N\mathbb{Z}$  and that for large N

$$(S_N f)\left(\frac{\pi}{2N}\right) \approx \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin u}{u} \, \mathrm{d}u = \frac{1}{2} + \int_0^{\pi/2} \frac{\sin u}{u(\pi - u)} \, \mathrm{d}u \ge 1.08$$

Hint for lower bound:  $\sin u \ge u - u^3/3!$ . Comment on the accuracy of partial Fourier series at discontinuities.

**12.** Set  $V = \{y \in C^2[a, b] : y(a) = y(b) = 0\}$  (i.e. Dirichlet boundary conditions) and let  $L = \frac{1}{w} \left[ -\frac{d}{dx} \left( p \frac{d}{dx} \right) + q \right]$  be a Sturm-Liouville operator with p, q, w smooth and p, w > 0 on [a, b]. Consider the Rayleigh quotient

$$R[y] = \frac{\int_a^b \left[ p\left(y'\right)^2 + qy^2 \right] \mathrm{d}x}{\int_a^b wy^2 \, \mathrm{d}x}, \quad y \in V$$

- (a) By considering  $\langle Ly, y \rangle_w$ , show that if  $y \in V$  satisfies  $Ly = \lambda y$  then  $\lambda = R[y]$ .
- (b) Let  $\lambda_1 = \inf_{y \in V \setminus \{0\}} R[y]$  and suppose that there exists a  $y_1 \in V$  such that  $R[y_1] = \lambda_1$ . If we set

$$F(\epsilon) = R[y_1 + \epsilon\eta],$$

where  $\eta \in V$ , explain why F'(0) = 0. Hence show that  $Ly_1 = \lambda_1 y_1$ . Comment on this result in relation to finding the smallest eigenvalue of L. How might you try to find the second smallest?

(c) Take [a,b] = [0,1] and  $L = -d^2/dx^2$ . Compute R[y] where y(x) = x(1-x) and deduce  $\lambda_{\min} = \pi^2 \leq 10$ .

<sup>&</sup>lt;sup>1</sup>You may assume that if  $\int_a^b f\eta \, dx = 0$  for all  $\eta \in V$  then f = 0.