METHODS — EXAMPLES II

Laplace's equation

1. *Cartesian coordinates*. Show that the solution of $\nabla^2\phi = 0$ in the region $0 < x < a,\ 0 < y < b,\ 0 < z < c$, with $\phi = 1$ on the surface $z = 0$ and $\phi = 0$ on all the other surfaces is

$$\phi = \frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\sinh[\ell(c-z)]\sin[(2p+1)\pi x/a]\sin[(2q+1)\pi y/b]}{(2p+1)(2q+1)\sinh c\ell},$$

where $\ell^2 = (2p+1)^2\pi^2/a^2 + (2q+1)^2\pi^2/b^2$. [*Hint:* You may find it useful to use the above form of the z-dependent part of the solution from the outset.] Discuss the behaviour of the solutions as $c \to \infty$.

2. *Plane polar coordinates*. The potential ϕ satisfies Laplace's equation in the unit circle $r < 1$ with boundary condition

$$\phi(r=1,\theta) = \begin{cases} \pi/2\,, & 0 \le \theta < \pi\,. \\ -\pi/2\,, & \pi \le \theta < 2\pi\,. \end{cases}$$

Using the method of separation of variables show that

$$\phi(r,\theta) = 2\sum_{n\ odd}\frac{r^n\sin n\theta}{n}\,.$$

Sum the series using the substitution $z = re^{i\theta}$. [Your solution can then be interpreted geometrically as the angle between the lines to the two points on the boundary where the data jumps.]

3. *Spherical polar coordinates*. The electrostatic potential in a charge-free region satisfies Laplace's equation. Find the potential inside a spherical region bounded by two (conducting) hemispheres upon which the potential takes the values $\pm V$ respectively. [*Hint:* Note that $P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z)$ and $\int_{-1}^{1} P_m(z)P_n(z)dz = \frac{2}{2n+1}\delta_{mn}$.]

Legendre polynomials

4. *Eigenfunction derivatives*. If Y_m and Y_n are real eigenfunctions of the Sturm-Liouville equation

$$\frac{d}{dx}(p(x)\frac{dy}{dx}) + (\lambda - q(x))y = 0\,,\ (a < x < b)\,,\ \text{ satisfying the normalisation condition } \int_a^b Y_m^2dx = \int_a^b Y_n^2dx = 1\,,$$

show that (under suitable boundary conditions)

$$\int_a^b (pY'_m\ Y'_n + qY_mY_n)dx = \lambda_m\delta_{mn} \quad \text{(no summation)}.$$

With P_n a Legendre polynomial, use this result to evaluate $\displaystyle\int_{-1}^{1}(1-x^2)P'_m(x)P'_n(x)dx\,.$

5. *Legendre polynomials and Rodrigues' formula**. Define $q_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n(x^2-1)^n$ for positive integers n.

(a) Show (i) q_n is a polynomial of degree n;
(ii) $q_n(1) = 1$ for all n;
(iii) q_n satisfies Legendre's equation.

(b) Hence, deduce that (i) $q_n = P_n(x)$;
(ii) $\int_{-1}^{1}[P_n(x)]^2dx = 2/(2n+1)$ (i.e. normalisation);
(iii) $\int_{-1}^{1} x^mP_n(x)dx = 0$ if $m < n$ (i.e. orthogonality).

[*Hint:* For (a)(iii) show $u_n = (x^2-1)^n$ satisfies $(x^2-1)u'_n - 2nxu_n = 0$ and differentiate further. For (b) integrate by parts.] Note that analogous generating functions, normalisations and recurrence relations are available for other orthogonal polynomials and are tabulated in (for example) the Digital Library of Mathematical Functions at `dlmf.nist.gov`.

The one-dimensional wave equation

6. *Modes on a string*. A uniform string of mass per unit length μ and tension T undergoes small transverse vibrations of amplitude $y(x,t)$. The string is fixed at $x=0$ and $x=\ell$, and satisfies the initial conditions

$$y(x,0)=0\,,\qquad y_t(x,0)=\frac{4V}{\ell^2}x(\ell-x)\,,\quad \text{for}\;\;0<x<\ell\,,$$

where $y_t\equiv\partial y/\partial t$. Using the fact that $y(x,t)$ is a solution of the wave equation, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time t. By comparison with the initial energy of the string show that

$$\sum_{\text{n odd}}\frac{1}{n^6}=\frac{\pi^6}{960}\,.$$

7. *Damped string motion*. (i) A uniform stretched string of length ℓ, mass per unit length μ and tension $T=\mu c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force per unit length being $-2k\mu y_t$, where y is the transverse displacement and the constant $k=\pi c/\ell$. Show that the equation of motion of the string is

$$c^2\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial t^2}+2k\frac{\partial y}{\partial t}\,,$$

and find $y(x,t)$ given that $y(x,0)=A\sin(\pi x/\ell)$ and $y_t(x,0)=0$.

(ii) If an extra transverse force $F_0\sin(\pi x/\ell)\cos(\pi ct/\ell)$ per unit length acts on the string, find the associated particular integral. Discuss the behaviour of the full solution as $t\to\infty$.

8. *Wave reflection and transmission*. A string of uniform density is stretched along the x-axis under tension τ and undergoes small transverse oscillations in the (x,y) plane so that its displacement $y(x,t)$ satisfies

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\,, \qquad (*)$$

where c is a constant.

(i) Show that if a mass M is fixed to the string at $x=0$ then its equation of motion can be written

$$M\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=0}=\tau\left[\frac{\partial y}{\partial x}\right]_{x=0_-}^{x=0_+}\,.$$

(ii) Suppose that a wave $\exp[i\omega(t-x/c)]$ is incident from $x=-\infty$. Obtain the amplitudes and phases of the reflected and transmitted waves, and comment on their values when $\lambda=M\omega c/\tau$ is large or small.

9. *Impulsive force on a string*. The displacement $y(x,t)$ of a uniform string stretched between the points $x=0,\ell$ satisfies the wave equation $(*)$ given above with the boundary conditions, $y(0,t)=y(\ell,t)=0$. For $t<0$ the string oscillates in its fundamental mode and $y(x,0)=0$. A musician strikes the midpoint of the string impulsively at time $t=0$ so that the change in $\partial y/\partial t$ at $t=0$ is $\lambda\delta(x-\frac{1}{2}\ell)$. Find $y(x,t)$ for $t>0$.

The heat equation

10. *Diffusion in a disc & Bessel functions**. Consider the unit disc, with initial temperature distribution $\psi_0(r,\theta)$. Require the boundary of the disc to be held at (wlog) zero temperature $\psi(1,\theta,t)=0$ for all $t>0$. By assuming that the temperature satisfies the diffusion equation in the disc (with unit diffusion coefficient) show that the solution is

$$\psi=\sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}c_{nk}J_n(j_{nk}r)\exp[in\theta-j_{nk}^2t],$$

where j_{nk} is the k^{th} smallest (positive) zero of the n^{th} order Bessel function of the first kind, (i.e. $J_n(j_{nk})=0$) and present an appropriate expression for c_{nk}. You may assume that (generalising Q8 from Sheet 1 to arbitrary positive integers n):

$$\int_0^1 J_n(j_{nk}r)J_n(j_{nl}r)r\,dr=\frac{\delta_{kl}[J_n'(j_{nk})]^2}{2}=\frac{\delta_{kl}J_{n+1}^2(j_{nk})}{2}.$$

Hence deduce the admissible forms for the initial conditions $\psi_0(r,\theta)$ so that the ratio $\Psi(r,\theta,t)=\psi(r,\theta,t)/\psi_0(r,\theta)$ is a function of time alone. Suppose now that $\psi_0=\Psi_0$ for all $r<1$. Show that the only non-zero coefficients have $n=0$, and are equal to

$$c_{0k}=\frac{2\Psi_0}{j_{0k}J_1(j_{0k})}.$$

What is the spatial structure of the temperature distribution as $t\to\infty$?

[*The recursion relations* $[z^{-\nu}J_\nu(z)]'=-z^{-\nu}J_{\nu+1}(z)$ *and* $[z^{\nu+1}J_{\nu+1}(z)]'=z^{\nu+1}J_\nu(z)$ *may be useful.*]

[†]If you find any errors in the Methods Examples sheets, please inform your supervisor or email epss@damtp.cam.ac.uk.