**Mathematical Tripos Part IB**
**Methods, Example Sheet 2**

**Comments and corrections to acla2@damtp.cam.ac.uk. Sheet with commentary available to supervisors.**

1. The function $\varphi = \varphi(x, y, z)$ satisfies the Laplace equation $\Delta \varphi = 0$ on the cuboid $(x, y, z) \in (0, a) \times (0, b) \times (0, c)$, such that $\varphi = 1$ on the side $z = 0$ and $\varphi = 0$ on all other sides. Show that

$$\varphi(x, y, z) = \frac{16}{\pi^2} \sum_{p,q=0}^{\infty} \frac{\sin[(\varphi_{p,q}(c-z)\sin[(2p+1)\pi x/a]\sin[(2q+1)\pi y/b]}{(2p+1)(2q+1)\sinh(\varphi_{p,q})}$$

where $\varphi_{p,q} = (2p+1)^2\pi^2/a^2 + (2q+1)^2\pi^2/b^2$. Discuss the behaviour of the solution as $c \to \infty$.

2. The function $\varphi = \varphi(r, \theta)$ satisfies the Laplace equation $\Delta \varphi = 0$ on the unit disc $(r, \theta) \in [0, 1) \times [0, 2\pi)$ such that $\varphi(1, \theta) = \pi/2$ on $0 \leq \theta < \pi$ and $\varphi(1, \theta) = -\pi/2$ on $\pi \leq \theta < 2\pi$. Show that

$$\varphi(r, \theta) = 2 \sum_{n \text{ odd}} \frac{r^n \sin(n\theta)}{n}.$$

Sum the series using the substitution $z = r\cos \theta$. Your solution can then be interpreted geometrically as the angle between the linear to the two points on the boundary where the data jumps.

3. The function $\varphi = \varphi(r, \theta)$ satisfies the Laplace equation $\Delta \varphi = 0$ on the unit ball $(r, \theta, \phi) \in [0, 1) \times [0, \pi] \times [0, 2\pi)$ such that $\varphi(1, \theta, \phi) = \pi/2$ on $0 \leq \theta < \pi/2$ and $\varphi(1, \theta, \phi) = -\pi/2$ on $\pi/2 \leq \theta \leq \pi$. Show that

$$\varphi(r, \theta, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta)$$

where $a_n$ are constants you should determine in terms of the Legendre polynomials. It will be useful to note that $P_{n+1}'(z) - P_{n-1}'(z) = (2n+1)P_n(z)$ and $\int_1^1 P_n(z)P_m(z)dz = 2\delta_{mn}/(2n+1)$.

4. A uniform string of mass per unit length $\mu$ and tension $\tau$ undergoes small transverse vibrations of amplitude $y = y(x, t)$. The string is fixed at $x = 0$ and $x = L$ and satisfies the initial conditions

$$y(x, 0) = 0, \quad \frac{\partial y}{\partial t}(x, 0) = \frac{4V}{L^2}x(L-x) \quad \text{for } 0 < x < L.$$

Using the fact that $y$ satisfies the wave equation with speed $c$ where $c^2 = \tau/\mu$, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time $t$. Hence show that

$$\sum_{n \text{ odd}} \frac{1}{n^6} = \frac{\pi^6}{960}.$$

5. The displacement $y = y(x, t)$ of a uniform string stretched between $x = 0$ and $x = L$ satisfies the wave equation with the boundary conditions $y(0, t) = y(L, t) = 0$. For $t < 0$ the string oscillates in the fundamental mode $y(x, t) = A \sin(\pi x/L) \sin(\pi ct/L)$. A musician strikes the midpoint of the string impulsively at time $t = 0$ so that the change in $\partial y/\partial t$ at $t = 0$ is $\lambda \delta(x - 1/2)$. Find $y = y(x, t)$ for $t > 0$.

6. Consider a uniform stretched string of length $L$, mass per unit length $\mu$, tension $\tau = \mu c^2$ and ends fixed.

   (i) The string undergoes transverse oscillations in a resistive medium that produces a resistive force per unit length of $-2k\gamma y$, where $y = y(x, t)$ is the transverse displacement and $k = \pi c/L$. Derive the equation of motion

   $$\frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + \frac{2k}{c^2} \frac{\partial y}{\partial t}.$$

   Find $y - y(x, t)$ if $y(x, 0) = A \sin(\pi x/L)$ and $y_t(x, 0) = 0$.

   (ii) If an extra transverse force $F \sin(\pi x/L) \cos(\pi ct/L)$ per unit length is applied to the string, find the associated particular integral. Discuss the behaviour of the full solution as $t \to \infty$.  

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7. A string of uniform density is stretched along the x-axis under tension \( \tau \). It undergoes small transverse oscillations so that the displacement \( y = y(x, t) \) satisfies the wave equation.

(i) Show that if a mass \( M \) is fixed to the string at \( x = 0 \) then its equation of motion can be written

\[
M \frac{\partial^2 y}{\partial t^2} \bigg|_{x=0} = \tau \left[ \frac{\partial y}{\partial x} \right]_{x=0}^+.
\]  

(ii) A wave of the form \( (x, t) \mapsto \exp[i \omega(t - x/c)] \) is incident from \( x \to -\infty \) giving rise to a solution of the form

\[
y(x, t) = \begin{cases} 
    e^{i \omega(t-x/c)} + R e^{i \omega(t+x/c)}, & x < 0, \\
    T e^{i \omega(t-x/c)}, & x > 0.
\end{cases}
\]

Using (i) and an appropriate continuity condition at \( x = 0 \), find expressions for \( T = T(\lambda) \) and \( R = R(\lambda) \) where \( \lambda = M \omega c/\tau \). Discuss the limiting behaviour of \( R \) and \( T \) when \( \lambda \) is large or small.

8. Here we solve the heat equation on an interval with non-zero boundary data. Let \( \varphi = \varphi(x, t) \) satisfy

\[
\begin{cases}
    \varphi_t - \kappa \varphi_{xx} = 0, & (x, t) \in (0, 1) \times (0, \infty), \\
    \varphi(x, 0) = x^2, & x \in [0, 1), \\
    \varphi(0, t) = 0, & t > 0, \\
    \varphi(1, t) = 1, & t > 0.
\end{cases}
\]

By considering a suitable function of the form \( \Phi(x, t) = \varphi(x, t) - (Ax + B) \) with \( A, B \) constant, reduce the problem to one for \( \Phi \) with homogeneous boundary data. Hence find \( \varphi(x, t) \) and discuss its behaviour as \( t \to \infty \).

**Additional problems**

*These questions should not be attempted at the expense of earlier ones.*

9. Let \( f = f(\theta) \) be 2\( \pi \)-periodic function and consider the periodic initial value problem for the heat equation \( \varphi_t = \varphi_{\theta\theta} \) with \( \varphi(\theta, 0) = f(\theta) \) and \( \varphi(\theta + 2\pi, t) = \varphi(\theta, t) \) for each \( (\theta, t) \). Using an appropriate Fourier series, solve for \( \varphi \) and write it in the form \( \varphi(\theta, t) = \int_0^{2\pi} \vartheta_t(\theta - \phi) f(\phi) \, d\phi \) where \( \vartheta_t(\theta) \) is a function you should determine.

10. Let \( \Omega \subseteq \mathbb{R}^3 \) be a bounded domain and \( (x, t) \in \Omega \times (0, \infty) \). We will be concerned with the following initial-boundary problems for the heat and wave equations, respectively:

\[
(A) \quad \begin{cases}
    \varphi_t - \kappa \Delta \varphi = 0, & \text{in } \Omega \times (0, \infty) \\
    \varphi = f, & \text{on } \Omega \times \{t = 0\} \\
    \varphi = 0, & \text{on } \partial \Omega \times (0, \infty)
\end{cases} \quad \text{and} \quad \begin{cases}
    \varphi_{tt} - \Delta \varphi = 0, & \text{in } \Omega \times (0, \infty) \\
    \varphi = g, & \text{on } \Omega \times \{t = 0\} \\
    \varphi_t = h, & \text{on } \Omega \times \{t = 0\} \\
    \varphi = 0, & \text{on } \partial \Omega \times (0, \infty)
\end{cases}
\]

You may assume the following: there is a collection \( \{(\psi_n, \lambda_n)\}_{n=1}^\infty \) of real eigenfunction-eigenvalue pairs such that (a) \(-\Delta \psi_n = \lambda_n \psi_n \) in \( \Omega \); (b) \( \psi_n = 0 \) on \( \partial \Omega \); (c) each eigenvalue has finite multiplicity; (d) \( \{\psi_n\} \) are complete on \( \Omega \). The latter means for \( f : \Omega \to \mathbb{R} \) satisfying \( f = 0 \) on \( \partial \Omega \) we can write \( f = \sum_n \alpha_n \psi_n \) for some \( \{\alpha_n\} \).

(i) Show that \( \lambda_n > 0 \) for each \( n \) and \( \int_\Omega \psi_n \psi_m \, dV = 0 \) for \( \lambda_n \neq \lambda_m \).

(ii) Explain why we can assume, without loss of generality, that \( \int_\Omega \psi_n \, dV = 0 \) for \( n \neq m \).

(iii) Using separation of variables, show that the solution to \((A)\) is given by

\[
\varphi(x, t) = \sum_{n=1}^\infty \alpha_n e^{-\lambda_n \kappa t} \psi_n(x) \quad \text{where} \quad \alpha_n = \frac{\int_\Omega f \psi_n \, dV}{\int_\Omega \psi_n^2 \, dV}.
\]

Explain why this might be formally interpreted as \( \varphi(x, t) = e^{\kappa t} \varphi(0, x) \) where \( e^{\kappa t} = \sum_{n=0}^\infty \frac{(\kappa t)^n}{n!} \Delta^n \).

(iv) Solve \((B)\), again using separation of variables. Relate your answer to the formal expression

\[
\varphi(x, t) = \frac{\sin(c \sqrt{-\Delta})}{c \sqrt{-\Delta}} \varphi_t(x, 0) + \cos(c \sqrt{-\Delta}) \varphi(x, 0).
\]