Mathematical Tripos Part IB Michaelmas 2023
Methods, Example Sheet 2 Dr A.C.L. Ashton

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1. The function $\varphi = \varphi(x, y, z)$ satisfies the Laplace equation $\Delta\varphi = 0$ on the cuboid $(x, y, z) \in (0, a)\times(0, b)\times(0, c)$, such that $\varphi = 1$ on the side $z = 0$ and $\varphi = 0$ on all other sides. Show that

$$\varphi(x, y, z) = \frac{16}{\pi^2}\sum_{p,q=0}^{\infty}\frac{\sinh[\ell_{p,q}(c-z)]\sin[(2p+1)\pi x/a]\sin[(2q+1)\pi y/b]}{(2p+1)(2q+1)\sinh(c\ell_{p,q})}$$

where $\ell_{p,q}^2 = (2p+1)^2\pi^2/a^2 + (2q+1)^2\pi^2/b^2$. Discuss the behaviour of the solution as $c \to \infty$.

2. The function $\varphi = \varphi(r, \theta)$ satisfies the Laplace equation $\Delta\varphi = 0$ on the unit disc $(r, \theta) \in [0, 1) \times [0, 2\pi)$ such that $\varphi(1, \theta) = \pi/2$ on $0 \le \theta < \pi$ and $\varphi(1, \theta) = -\pi/2$ on $\pi \le \theta < 2\pi$. Show that

$$\varphi(r, \theta) = 2\sum_{n \text{ odd}}\frac{r^n\sin(n\theta)}{n}.$$

Sum the series using the substitution $z = re^{\mathrm{i}\theta}$. Your solution can then be interpreted geometrically as the angle between the linear to the two points on the boundary where the data jumps.

3. The function $\varphi = \varphi(r, \theta)$ satisfies the Laplace equation $\Delta\varphi = 0$ on the unit ball $(r, \theta, \phi) \in [0, 1)\times[0, \pi]\times[0, 2\pi)$ such that $\varphi(1, \theta) = 1$ on $0 \le \theta < \pi/2$ and $\varphi(1, \theta) = -1$ on $\pi/2 \le \theta \le \pi$. Show that

$$\varphi(r, \theta) = \sum_{n=0}^{\infty}\alpha_n r^n P_n(\cos\theta)$$

where α_n are constants you should determine in terms of the Legendre polynomials. It will be useful to note that $P'_{n+1}(z) - P'_{n-1}(z) = (2n + 1)P_n(z)$ and $\int_{-1}^{1} P_n(z)P_m(z)\,\mathrm{d}z = 2\delta_{mn}/(2n + 1)$.

4. A uniform string of mass per unit length μ and tension τ undergoes small transverse vibrations of amplitude $y = y(x, t)$. The string is fixed at $x = 0$ and $x = L$ and satisfies the initial conditions

$$y(x, 0) = 0, \quad \frac{\partial y}{\partial t}(x, 0) = \frac{4V}{L^2}x(L - x) \quad \text{for } 0 < x < L.$$

Using the fact that y satisfies the wave equation with speed c where $c^2 = \tau/\mu$, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time t. Hence show that

$$\sum_{n \text{ odd}}\frac{1}{n^6} = \frac{\pi^6}{960}.$$

5. The displacement $y = y(x, t)$ of a uniform string stretched between $x = 0$ and $x = L$ satisfies the wave equation with the boundary conditions $y(0, t) = y(L, t) = 0$. For $t < 0$ the string oscillates in the fundamental mode $y(x, t) = A\sin(\pi x/L)\sin(\pi ct/L)$. A musician strikes the midpoint of the string impulsively at time $t = 0$ so that the change in $\partial y/\partial t$ at $t = 0$ is $\lambda\delta(x - \frac{1}{2}L)$. Find $y = y(x, t)$ for $t > 0$.

6. Consider a uniform stretched string of length L, mass per unit length μ, tension $\tau = \mu c^2$ and ends fixed.
(i) The string undergoes transverse oscillations in a resistive medium that produces a resistive force per unit length of $-2k\mu y_t$, where $y = y(x, t)$ is the transverse displacement and $k = \pi c/L$. Derive the equation of motion

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \frac{2k}{c^2}\frac{\partial y}{\partial t}.$$

Find $y = y(x, t)$ if $y(x, 0) = A\sin(\pi x/L)$ and $y_t(x, 0) = 0$.
(ii) If an extra transverse force $F\sin(\pi x/L)\cos(\pi ct/L)$ per unit length is applied to the string, find the associated particular integral. Discuss the behaviour of the full solution as $t \to \infty$.

7. A string of uniform density is stretched along the x-axis under tension τ. It undergoes small transverse oscillations so that the displacement $y = y(x,t)$ satisfies the wave equation.
(i) Show that if a mass M is fixed to the string at $x = 0$ then its equation of motion can be written

$$M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=0} = \tau \left[\frac{\partial y}{\partial x}\right]_{x=0^-}^{x=0^+}. \qquad (\star)$$

(ii) A wave of the form $(x,t) \mapsto \exp[\mathrm{i}\omega(t - x/c)]$ is incident from $x \to -\infty$ giving rise to a solution of the form

$$y(x,t) = \begin{cases} e^{\mathrm{i}\omega(t-x/c)} + Re^{\mathrm{i}\omega(t+x/c)}, & x < 0, \\ Te^{\mathrm{i}\omega(t-x/c)}, & x > 0. \end{cases}$$

Using $(\star)$ and an appropriate continuity condition at $x = 0$, find expressions for $T = T(\lambda)$ and $R = R(\lambda)$ where $\lambda = M\omega c/\tau$. Discuss the limiting behaviour of R and T when λ is large or small.

8. Here we solve the heat equation on an interval with *non-zero* boundary data. Let $\varphi = \varphi(x,t)$ satisfy

$$\begin{cases} \varphi_t - \kappa\varphi_{xx} = 0, & (x,t) \in (0,1) \times (0,\infty), \\ \varphi(x,0) = x^2, & x \in (0,1), \\ \varphi(0,t) = 0, & t > 0, \\ \varphi(1,t) = 1, & t > 0. \end{cases}$$

By considering a suitable function of the form $\Phi(x,t) = \varphi(x,t) - (Ax + B)$ with A, B constant, reduce the problem to one for Φ with homogeneous boundary data. Hence find $\varphi(x,t)$ and discuss its behaviour as $t \to \infty$.

Additional problems

*These questions should **not** be attempted at the expense of earlier ones.*

9. Let $f = f(\theta)$ be 2π-periodic function and consider the periodic initial value problem for the heat equation $\varphi_t = \varphi_{\theta\theta}$ with $\varphi(\theta,0) = f(\theta)$ and $\varphi(\theta + 2\pi, t) = \varphi(\theta,t)$ for each (θ,t). Using an appropriate Fourier series, solve for φ and write it in the form $\varphi(\theta,t) = \int_0^{2\pi} \vartheta_t(\theta - \phi) f(\phi)\,\mathrm{d}\phi$ where $\vartheta_t(\theta)$ is a function you should determine.

10. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain and $(\mathbf{x},t) \in \Omega \times (0,\infty)$. We will be concerned with the following initial-boundary value problems for the heat and wave equations, respectively:

$$(A)\ \begin{cases} \varphi_t - \kappa\Delta\varphi = 0, & \text{in } \Omega \times (0,\infty) \\ \varphi = f, & \text{on } \Omega \times \{t = 0\} \\ \varphi = 0, & \text{on } \partial\Omega \times [0,\infty) \end{cases} \qquad (B)\ \begin{cases} \varphi_{tt} - c^2\Delta\varphi = 0, & \text{in } \Omega \times (0,\infty) \\ \varphi = g, & \text{on } \Omega \times \{t = 0\} \\ \varphi_t = h, & \text{on } \Omega \times \{t = 0\} \\ \varphi = 0, & \text{on } \partial\Omega \times [0,\infty) \end{cases}$$

You may *assume* the following: there is a collection $\{(\psi_n, \lambda_n)\}_{n=1}^\infty$ of real eigenfunction-eigenvalue pairs such that (a) $-\Delta\psi_n = \lambda_n\psi_n$ in Ω; (b) $\psi_n = 0$ on $\partial\Omega$; (c) each eigenvalue has finite multiplicity; (d) $\{\psi_n\}$ are complete on Ω. The latter means for $f : \Omega \to \mathbf{R}$ satisfying $f = 0$ on $\partial\Omega$ we can write $f = \sum_n \alpha_n\psi_n$ for some $\{\alpha_n\}$.
(i) Show that $\lambda_n > 0$ for each n and $\int_\Omega \psi_n\psi_m\,\mathrm{d}V = 0$ for $\lambda_n \neq \lambda_m$.
(ii) Explain why we can assume, without loss of generality, that $\int_\Omega \psi_n\psi_m\,\mathrm{d}V = 0$ for $n \neq m$.
(iii) Using separation of variables, show that the solution to (A) is given by

$$\varphi(\mathbf{x},t) = \sum_{n=1}^\infty \alpha_n e^{-\lambda_n\kappa t}\psi_n(\mathbf{x}) \quad \text{where} \quad \alpha_n = \frac{\int_\Omega f\psi_n\,\mathrm{d}V}{\int_\Omega \psi_n^2\,\mathrm{d}V}.$$

Explain why this might be formally interpreted as $\varphi(\mathbf{x},t) = e^{\kappa t\Delta}\varphi(\mathbf{x},0)$ where $e^{\kappa t\Delta} = \sum_{p=0}^\infty \frac{(\kappa t)^p}{p!}\Delta^p$.
(iv) Solve (B), again using separation of variables. Relate your answer to the formal expression

$$\varphi(\mathbf{x},t) = \frac{\sin\left(ct\sqrt{-\Delta}\right)}{c\sqrt{-\Delta}}\varphi_t(\mathbf{x},0) + \cos\left(ct\sqrt{-\Delta}\right)\varphi(\mathbf{x},0).$$