

METHODS — EXAMPLES II

The one-dimensional wave equation

1. *Modes on a string.* A uniform string of line density μ and tension T undergoes small transverse vibrations of amplitude $y(x, t)$. The string is fixed at $x = 0$ and $x = \ell$, and satisfies the initial conditions

$$y(x, 0) = 0, \quad y_t(x, 0) = \frac{4V}{\ell^2} x(\ell - x), \quad \text{for } 0 < x < \ell,$$

where $y_t \equiv \partial y / \partial t$. Using the fact that $y(x, t)$ is a solution of the wave equation, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time t . By comparison with the initial energy of the string show that

$$\sum_{n \text{ odd}} \frac{1}{n^6} = \frac{\pi^6}{960}.$$

2. *Damped string motion.* (i) A uniform stretched string of length ℓ , density per unit length μ and tension $T = \mu c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force on unit length being $-2k\mu y_t$, where y is the transverse displacement and the constant $k = \pi c / \ell$. Show that the equation of motion of the string is

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t},$$

and find $y(x, t)$ given that $y(x, 0) = A \sin(\pi x / \ell)$ and $y_t(x, 0) = 0$.

(ii) If an extra transverse force $F_0 \sin(\pi x / \ell) \cos(\pi ct / \ell)$ per unit length acts on the string, find the resulting forced oscillation. [*Hint:* You may find it useful to find the forced component of the solution which dominates as $t \rightarrow \infty$.]

3. *Wave reflection and transmission.* A string of uniform density is stretched along the x -axis under tension T and undergoes small transverse oscillations in the (x, y) plane so that its displacement $y(x, t)$ satisfies

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{*}$$

where c is a constant.

(i) Show that if a mass M is fixed to the string at $x = 0$ then its equation of motion can be written

$$M \frac{d^2 y}{dt^2} \Big|_{x=0} = T \left[\frac{\partial y}{\partial x} \right]_{x=0+}^{x=0-}.$$

(ii) Suppose that a wave $\exp[i\omega(t - x/c)]$ is incident from $x = -\infty$. Obtain the amplitudes and phases of the reflected and transmitted waves, and comment on their values when $\lambda = M\omega c / T$ is large or small.

4. *Impulsive force on a string.* The displacement $y(x, t)$ of a uniform string stretched between the points $x = 0, \ell$ satisfies the wave equation (*) given above with the boundary conditions, $y(0, t) = y(\ell, t) = 0$. For $t < 0$ the string oscillates in its fundamental mode and $y(x, 0) = 0$. A musician strikes the midpoint of the string impulsively at time $t = 0$ so that the change in $\partial y / \partial t$ at $t = 0$ is $\lambda \delta(x - \frac{1}{2}\ell)$. Find $y(x, t)$ for $t > 0$.

Laplace's equation

5. *Cartesian coordinates.* Show that the solution of $\nabla^2 \phi = 0$ in the region $0 < x < a, 0 < y < b, 0 < z < c$, with $\phi = 1$ on the surface $z = 0$ and $\phi = 0$ on all the other surfaces is

$$\phi = \frac{16}{\pi^2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\sinh[\ell(c - z)] \sin[(2p + 1)\pi x / a] \sin[(2q + 1)\pi y / b]}{(2p + 1)(2q + 1) \sinh c\ell},$$

where $\ell^2 = (2p + 1)^2 \pi^2 / a^2 + (2q + 1)^2 \pi^2 / b^2$. [*Hint:* You may find it useful to use the above form of the z -dependent part of the solution from the outset.] Discuss the behaviour of the solutions as $c \rightarrow \infty$.

6. Plane polar coordinates. The potential ϕ satisfies Laplace's equation in the unit circle $r < 1$ with boundary condition

$$\phi(r = 1, \theta) = \begin{cases} \pi/2, & 0 \leq \theta < \pi. \\ -\pi/2, & \pi \leq \theta < 2\pi. \end{cases}$$

Using the method of separation of variables show that

$$\phi(r, \theta) = 2 \sum_{n \text{ odd}} \frac{r^n \sin n\theta}{n}.$$

Sum the series using the substitution $z = re^{i\theta}$. [Your solution can then be interpreted geometrically as the angle between the lines to the two points on the boundary where the data jumps.]

7. Spherical polar coordinates. The electrostatic potential in a charge-free region satisfies Laplace's equation. Find the potential inside a spherical region bounded by two (conducting) hemispheres upon which the potential takes the values $\pm V$ respectively. [Hint: Note that $P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z)$ and $\int_{-1}^1 P_m(z)P_n(z)dz = \frac{2}{2n+1}\delta_{mn}$.]

Legendre polynomials

8. Eigenfunction derivatives. If y_m and y_n are real eigenfunctions of the Sturm-Liouville equation

$$\frac{d}{dx}(p(x)\frac{dy}{dx}) + (\lambda - q(x))y = 0, \quad (a < x < b), \quad \text{satisfying the normalisation condition } \int_a^b y_m^2 dx = \int_a^b y_n^2 dx = 1,$$

show that (under suitable boundary conditions)

$$\int_a^b (py'_m y'_n + qy_m y_n) dx = \lambda_m \delta_{mn} \quad (\text{no summation}).$$

With P_n a Legendre polynomial, use this result to evaluate $\int_{-1}^1 (1-x^2)P'_m(x)P'_n(x)dx$.

9. Legendre polynomials and Rodrigues' formula. Define $q_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n (x^2-1)^n$ for positive integers n .

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| (a) Show (i) q_n is a polynomial of degree n ; | (b) Hence, deduce that (i) $q_n = P_n(x)$; |
| (ii) $q_n(1) = 1$ for all n ; | (ii) $\int_{-1}^1 [P_n(x)]^2 dx = 2/(2n+1)$ (i.e. normalisation); |
| (iii) q_n satisfies Legendre's equation. | (iii) $\int_{-1}^1 x^m P_n(x) dx = 0$ if $m < n$ (i.e. orthogonality). |

[Hint: For (a)(iii) show $u_n = (x^2-1)^n$ satisfies $(x^2-1)u'_n - 2nxu_n = 0$ and differentiate further. For (b) integrate by parts.] Note that analogous generating functions, normalisations and recurrence relations are available for other orthogonal polynomials and are tabulated in e.g. Abramowitz & Stegun.

The heat equation

10. Diffusion in a disc & Bessel functions. Consider the unit disc, with initial temperature distribution $\psi_0(r, \theta)$. Require the boundary of the disc to be held at (wlog) zero temperature $\psi(1, \theta, t) = 0$ for all $t > 0$. By assuming that the temperature satisfies the diffusion equation in the disc (with unit diffusion coefficient) show that the solution is

$$\psi = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} c_{nk} J_n(j_{nk}r) \exp[in\theta - j_{nk}^2 t],$$

where j_{nk} is the k^{th} smallest (positive) zero of the n^{th} order Bessel function of the first kind, (i.e. $J_n(j_{nk}) = 0$) and present an appropriate expression for c_{nk} , showing explicitly that

$$\int_0^1 J_n(j_{nk}r) J_n(j_{nl}r) r dr = \frac{\delta_{kl} [J'_n(j_{nk})]^2}{2} = \frac{\delta_{kl} J_{n+1}^2(j_{nk})}{2}.$$

Hence deduce the admissible forms for the initial conditions $\psi_0(r, \theta)$ so that the ratio $\Psi(r, \theta, t) = \psi(r, \theta, t)/\psi_0(r, \theta)$ is a function of time alone. Suppose now that $\psi_0 = \Psi_0$ for all $r < 1$. Show that the only non-zero coefficients have $n = 0$, and are equal to

$$c_{0k} = \frac{2\Psi_0}{j_{0k} J_1(j_{0k})}.$$

What is the spatial structure of the temperature distribution as $t \rightarrow \infty$?

[The recursion relations $[z^{-\nu} J_\nu(z)]' = -z^{-\nu} J_{\nu+1}(z)$ and $[z^{\nu+1} J_{\nu+1}(z)]' = z^{\nu+1} J_\nu(z)$ may be useful, as is the fact that Q8 of the first example sheet can be generalized straightforwardly to J_n for arbitrary positive integers n .]