METHODS — EXAMPLES II

The one-dimensional wave equation

1. Modes on a string. A uniform string of line density \( \mu \) and tension \( T = \mu c^2 \) undergoes small transverse vibrations of amplitude \( y(x,t) \). The string is fixed at \( x = 0 \) and \( x = \ell \), and satisfies the initial conditions

\[
y(x,0) = 0, \quad y_t(x,0) = \frac{4V}{\ell^2} x(\ell - x), \quad \text{for} \quad 0 < x < \ell,
\]

where \( y_t \equiv \partial y/\partial t \). Using the fact that \( y(x,t) \) is a solution of the wave equation, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time \( t \). By comparison with the initial energy of the string show that

\[
\sum_{n \text{ odd}} \frac{1}{n^6} = \frac{\pi^6}{960}.
\]

2. Damped string motion. (i) A uniform stretched string of length \( \ell \), density per unit length \( \mu \) and tension \( T = \mu c^2 \) is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force on unit length being \(-2k\mu y_t\), where \( y \) is the transverse displacement and the constant \( k = \pi c/\ell \). Show that the equation of motion of the string is

\[
c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t},
\]

and find \( y(x,t) \) given that \( y(x,0) = A \sin(\pi x/\ell) \) and \( y_t(x,0) = 0 \).
(ii) If an extra transverse force \( F_0 \sin(\pi x/\ell) \cos(\pi ct/\ell) \) per unit length acts on the string, find the resulting forced oscillation. [Hint: You may find it useful to guess a particular solution to combine with the general homogeneous solution that you probably derived in (i).]

3. Wave reflection and transmission. A string of uniform density is stretched along the \( x \)-axis under tension \( T \) and undergoes small transverse oscillations in the \((x,y)\) plane so that its displacement \( y(x,t) \) satisfies

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{(*)}
\]

where \( c \) is a constant.
(i) Show that if a mass \( M \) is fixed to the string at \( x = 0 \) then its equation of motion can be written

\[
M \frac{\partial^2 y}{\partial t^2} \bigg|_{x=0} = T \left[ \frac{\partial y}{\partial x} \right]_{x=0^+} - \left[ \frac{\partial y}{\partial x} \right]_{x=0^-}. \]

(ii) Suppose that a wave \( \exp[i \omega(t - x/c)] \) is incident from \( x = -\infty \). Obtain the amplitudes and phases of the reflected and transmitted waves, and comment on their values when \( \lambda = M \omega c/T \) is large or small.

4. Impulsive force on a string. The displacement \( y(x,t) \) of a uniform string stretched between the points \( x = 0, \ell \) satisfies the wave equation (*) given above with the boundary conditions, \( y(0,t) = y(\ell,t) = 0 \). For \( t < 0 \) the string oscillates in its fundamental mode and \( y(x,0) = 0 \). A musician strikes the midpoint of the string impulsively at time \( t = 0 \) so that the change in \( \partial y/\partial t \) at \( t = 0 \) is \( \lambda \delta(x - 1/2\ell) \). Find \( y(x,t) \) for \( t > 0 \).

Laplace’s equation

5. Cartesian coordinates. Show that the solution of \( \nabla^2 \phi = 0 \) in the region \( 0 < x < a, 0 < y < b, 0 < z < c \), with \( \phi = 1 \) on the surface \( z = 0 \) and \( \phi = 0 \) on all the other surfaces is

\[
\phi = \frac{16}{\pi^2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\sin[(2p + 1)\pi x/a] \sin[(2q + 1)\pi y/b]}{(2p + 1)(2q + 1)\sin \ell},
\]

where \( \ell^2 = (2p + 1)^2\pi^2/a^2 + (2q + 1)^2\pi^2/b^2 \). [Hint: You may find it useful to use the above form of the \( z \)-dependent part of the solution from the outset.] Discuss the behaviour of the solutions as \( c \to \infty \).
6. Plane polar coordinates. The potential \( \phi \) satisfies Laplace’s equation in the unit circle \( r < 1 \) with boundary condition

\[
\phi(r = 1, \theta) = \begin{cases} 
\pi/2, & 0 < \theta < \pi, \\
-\pi/2, & \pi < \theta < 2\pi.
\end{cases}
\]

Using the method of separation of variables show that the solution is

\[
\phi(r, \theta) = 2 \sum_{n \text{ odd}} \frac{r^n \sin n\theta}{n}.
\]

Sum the series using the substitution \( z = re^{i\theta} \). [Your solution can then be interpreted geometrically in terms of the angle between the lines to the two points on the boundary where the data jumps.]

Legendre polynomials

7. Eigenfunction derivatives. If \( y_m \) and \( y_n \) are real eigenfunctions of the Sturm-Liouville equation

\[
\frac{d}{dx} (p(x) \frac{dy}{dx}) + (\lambda - q(x))y = 0, \ (a < x < b),
\]

satisfying the normalisation condition \( \int_a^b y_m^2 dx = \int_a^b y_n^2 dx = 1 \), show that (under suitable boundary conditions)

\[
\int_a^b (py_m' y_n' + qy_m y_n) dx = \lambda_m \delta_{mn} \quad \text{(no summation)}.
\]

With \( P_n \) a Legendre polynomial, use this result to evaluate \( \int_{-1}^1 (1 - x^2) P'_m(x) P'_n(x) dx \).

8. Legendre polynomials and multipole moments. Show that \( 1/|r - k| \) obeys Laplace’s equation in three dimensions whenever \( r \neq k \). Taking \( k \) to be a unit vector in the \( z \)-direction, show that

\[
P_l(x) = \frac{1}{l!} \left. \frac{d^l}{dr^l} \frac{1}{\sqrt{1 - 2r \cos \theta + r^2}} \right|_{r=0}
\]

by expanding this solution of Laplace’s equation in the region \( r < 1 \). Use the integral

\[
\int_{-1}^1 \frac{1}{1 - 2rx + r^2} dx
\]

to show that the Legendre polynomials obey the normalization condition \( \int_{-1}^1 P_l(x)^2 dx = 2/(2l + 1) \). Show also that \( P_{l+1}'(x) - P_{l-1}'(x) = (2l + 1)P_l(x) \).

9. Spherical polar coordinates. You’ve just shown that the electrostatic potential in a charge-free region satisfies Laplace’s equation. Find the potential inside a spherical region bounded by two (conducting) hemispheres upon which the potential takes the values \( \pm V \) respectively. [Hint: Note that \( \int_{-1}^1 P_m(x)P_n(x) dx = \frac{2}{2m+1} \delta_{mn} \).]

The heat equation

10. Diffusion in a disc & Bessel functions. Consider the unit disc, with initial temperature distribution \( \psi_0(r, \theta) \). Require the boundary of the disc to be held at (wlog) zero temperature \( \psi(1, \theta, t) = 0 \) for all \( t > 0 \). By assuming that the temperature satisfies the diffusion equation in the disc (with unit diffusion coefficient) show that the solution is

\[
\psi = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} c_{nk} J_n(j_{nk} r) \exp[in\theta - j_{nk}^2 t],
\]

where \( j_{nk} \) is the \( k \)-th smallest (positive) zero of the \( n \)-th order Bessel function of the first kind, (i.e. \( J_n(j_{nk}) = 0 \)) and present an appropriate expression for \( c_{nk} \), showing explicitly that

\[
\int_0^1 J_n(j_{nk} r) J_n(j_{nt} r) r dr = \frac{\delta_{kl} J_{n+1}(j_{nk})^2}{2} = \frac{\delta_{kl} J_{n+1}(j_{nk})^2}{2}.
\]

Suppose now that the initial temperature \( \psi_0(r, \theta) = \Psi_0 \) is constant. Show that the only non-zero coefficients have \( n = 0 \), and are equal to

\[
c_{0k} = \frac{2\Psi_0}{j_{0k} J_{1}(j_{0k})}.
\]

What is the spatial structure of the temperature distribution as \( t \to \infty \)? [The recursion relations \( z^{-\nu} J_{\nu}(z) = -z^{-\nu} J_{\nu+1}(z) \) and \( z^{\nu+1} J_{\nu+1}(z) = z^{\nu+1} J_{\nu}(z) \) may be useful, as is the fact that Q8 of the first example sheet can be generalized straightforwardly to \( J_n \) for arbitrary positive integers \( n \).]