Green’s functions

1. **Initial value problem.** The reading \( \theta(t) \) of an ammeter satisfies

\[
\ddot{\theta} + 2p\dot{\theta} + (p^2 + q^2)\theta = f(t),
\]

where \( p, q \) are constants with \( p > 0 \). The ammeter is set so that \( \theta \) and \( \dot{\theta} \) are zero when \( t = 0 \). Assuming \( q \neq 0 \), show by constructing the Green’s function that

\[
\theta(t) = \frac{1}{q} \int_0^t e^{-p(t-\tau)} \sin[q(t-\tau)]f(\tau)d\tau.
\]

Derive the same result using Fourier transforms, showing that the transfer function for this system is

\[
\tilde{R}(\omega) = \frac{1}{2q} \left[ \frac{1}{i\omega + p - qi} - \frac{1}{i\omega + p + qi} \right].
\]

2. **Boundary value problem.** Obtain the Green’s function \( G(x, \xi) \) satisfying

\[
\frac{d^2 G}{dx^2} - \lambda^2 G = \delta(x - \xi), \quad 0 \leq x \leq 1, \quad 0 \leq \xi \leq 1,
\]

where \( \lambda \) is real, subject to the boundary conditions \( G(0, \xi) = G(1, \xi) = 0 \). Show that the solution to the equation

\[
\frac{d^2 y}{dx^2} - \lambda^2 y = f(x), \quad \text{subject to the same boundary conditions is}
\]

\[
y = -\frac{1}{\lambda \sinh \lambda} \left\{ \sinh \lambda x \int_x^1 f(\xi) \sinh \lambda(1 - \xi)d\xi + \sinh \lambda(1 - x) \int_0^x f(\xi) \sinh \lambda \xi d\xi \right\}.
\]

3. **Finite asymptotics.** A linear differential operator is defined by

\[
L_x y = -\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + y.
\]

By writing \( y = z/x \) or otherwise, find those solutions of \( L_x y = 0 \) which are either (a) bounded as \( x \to 0 \), or (b) bounded as \( x \to \infty \). Find the Green’s function \( G(x, a) \) satisfying

\[
L_x G(x, a) = \delta(x - a),
\]

and both conditions (a) and (b). Use \( G(x, a) \) to solve (subject to conditions (a) and (b))

\[
L_x y(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq R, \\ 0, & \text{for } x > R. \end{cases}
\]

Show that the solution has the form, for suitable constants \( A, B \)

\[
y(x) = \begin{cases} 1 + Ax^{-1}\sinh x, & \text{for } 0 \leq x \leq R, \\ Bx^{-1}e^{-x}, & \text{for } x > R. \end{cases}
\]

4. **Higher order initial value problem.** Show that the Green’s function for the initial value problem \( (\equiv \frac{d}{dt}) \)

\[
y''' + k^2 y'' = f(t), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0,
\]

is given by

\[
G(t, \tau) = \begin{cases} 0, & 0 < t < \tau, \\ k^{-2}(t-\tau) - k^{-3}\sin k(t-\tau), & t > \tau. \end{cases}
\]

Therefore, write down the integral form of the solution when \( f(t) = e^{-t} \) and verify that this integral satisfies the equation and the initial conditions.

[Hint: Make life easy by noting \( G(\tau, \tau) = 0 \) for an IVP Green’s function and so use the time invariance of the equation to take \( G(t, \tau) = f(t-\tau) \) for \( t > \tau \).]
The Dirac delta function

5. Delta function properties. The function \( \phi(x) \) is monotone increasing in \([a, b]\) and has a (simple) zero at \( x = c \) (i.e. \( \phi'(c) \neq 0 \)) where \( a < c < b \). Show that

\[
\int_a^b f(x)\delta(\phi(x))dx = \frac{f(c)}{|\phi'(c)|}.
\]

Show that the same formula applies if \( \phi(x) \) is monotone decreasing and hence derive a formula for general \( \phi(x) \) provided the zeros are simple. Deduce that \( \delta(at) = \delta(t)/|a| \) for \( a \neq 0 \). Also establish that

\[
\int_{-\infty}^{+\infty} |x|\delta(x^2 - a^2)dx = 1.
\]

6. Delta function derivative*. Show using polar coordinates that

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x^2 + y^2)\delta'(x^2 + y^2 - 1)\delta(x^2 - y^2)dx dy = f(1) - f'(1).
\]

Fourier transforms

7. Fourier transforms of functions of finite extent. Calculate the Fourier transforms of the following functions. All are non-zero only on the interval \( |x| < c \), and zero elsewhere.

\[
\begin{align*}
 f(x) &= 1, \\
 f(x) &= e^{iax}, \\
 f(x) &= \sin(ax), \\
 f(x) &= \cos(ax).
\end{align*}
\]

8. Functions with discontinuities. Let \( f(x) = e^{-x} \) for \( 0 < x < \infty \), and \( f(x) = 0 \) for \( x < 0 \). Show that

\[
\hat{f}(k) = \frac{1}{1 + k^2}.
\]

Show that the inverse Fourier transform of this Fourier transform \( \hat{f}(k) \) takes the value of \( 1/2 \) at \( x = 0 \). (This is a general property of Fourier transforms, analogously to Fourier series. Inversion for general \( x \) is really straightforward with Complex Methods.)

9. Fourier transform of Gaussians. By using differentiation with respect to wavenumber \( k \) and the shift property, calculate the Fourier transform of a Gaussian distribution with a peak at \( \mu \neq 0 \), i.e. \( f(x) = \exp[-n^2(x - \mu)^2] \).

Now let \( \mu = 0 \), and consider \( \delta_n(x) = (n/\sqrt{\pi})f(x) \). Sketch \( \delta_n(x) \) and \( \hat{\delta}_n(k) \) for small and large \( n \). What is \( \int_{-\infty}^{+\infty} \delta_n(x)dx \)? What is happening as \( n \to \infty \)?

10. Parseval’s relation for the discrete Fourier transform. Using the notation of the lecture notes, prove Parseval’s relation for the DFT:

\[
\sum_{m=0}^{N-1} |h(t_m)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |\hat{h}_d(f_n)|^2.
\]

11. Parseval’s relation continued. By considering the the Fourier transform of the function \( f(x) = \cos(x) \) for \( |x| < \pi/2 \) and \( f(x) = 0 \) for \( |x| \geq \pi/2 \), and the Fourier transform of its derivative, show that

\[
\int_{0}^{\infty} \frac{\pi^2 \cos^2 t}{(\pi^2 - t^2)^2} dt = \int_{0}^{\infty} \frac{t^2 \cos^2 t}{(\pi^2 - t^2)^2} dt = \frac{\pi}{4}.
\]

12. Laplace’s equation. Show that the inverse Fourier transform of the function

\[
\hat{f}(k) = \begin{cases} 
 e^k - e^{-k}, & |k| \leq 1, \\
 0, & |k| > 1,
\end{cases}
\]

is

\[
f(x) = \frac{2i}{\pi(1 + x^2)}(\cosh 1 \sin x - x \cos x \sinh 1).
\]

Determine, by using Fourier transforms, the solution of Laplace’s equation in the infinite strip \( 0 \leq y \leq 1 \), i.e.

\[
\nabla^2 \psi = 0; \quad -\infty < x < \infty, \quad 0 < y < 1,
\]

where \( \psi(x, 0) = f(x) \) the function given above, and \( \psi(x, 1) = 0 \) for \( -\infty < x < \infty \).

(This was a long tripos question (2004/4/II/15A) for Complex Methods on material now in the Methods schedule.)

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1 If you find any errors in the Methods Examples sheets, please inform your supervisor or email epss@damtp.cam.ac.uk.