Mathematical Tripos Part IB Michaelmas 2024
Methods, Example Sheet 3 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Let $\{\delta_n\}_{n=1}^{\infty}$ denote the sequence of functions that approximates the Dirac Delta, as seen in lectures. Recall the properties: $x \mapsto \delta_n(x)$ is smooth and non-negative; $\delta_n(x) = 0$ if $|nx| > 1$; and $\forall \epsilon > 0\ \exists N > 0$ such that $\forall n > N$ $\int_{-\epsilon}^{\epsilon} \delta_n(x)\,\mathrm{d}x = 1$. Using this definition, show that for any smooth function $f = f(x)$

$$\text{(i)} \quad \int \delta(x-a) f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int \delta_n(x-a) f(x)\,\mathrm{d}x = f(a).$$

For the rest of the sheet treat $\delta(x)$ as an ordinary function, as we do in lectures. Establish the results

$$\text{(ii)} \quad \int \delta'(x) f(x)\,\mathrm{d}x = -f'(0), \qquad \text{(iii)} \quad \lim_{h\to 0} \int \left[\frac{\delta(x+h)-\delta(x)}{h}\right] f(x)\,\mathrm{d}x = -f'(0).$$

Deduce that $\lim_{h\to 0} \frac{1}{h}\left[\delta(x+h) - \delta(x)\right] = \delta'(x)$.

2. Let $\phi : [a,b] \to \mathbf{R}$ be a monotone increasing with a simple zero $\phi(c) = 0$, $\phi'(c) \neq 0$ for some $c \in (a,b)$. Show that

$$\int_a^b f(x)\delta[\phi(x)]\,\mathrm{d}x = \frac{f(c)}{|\phi'(c)|}.$$

Show that the same formula holds if ϕ is monotone decreasing, and hence derive a formula for general ϕ, provided the zeros are simple. Deduce that $\delta(at) = \delta(t)/|a|$ for $a \neq 0$ and also establish the identity

$$\delta\left(x^2 - y^2\right) = \frac{\delta(x-y) + \delta(x+y)}{2|y|}.$$

3. By constructing an appropriate Green's function, find the general solution to boundary value problem

$$\begin{cases} y'' - 2y' + y = f(x), \quad 0 < x < 1 \\ y(0) = y(1) = 0, \end{cases}$$

4. Let $\lambda \in \mathbf{R} \setminus \{0\}$ be given. Obtain the Dirichlet Green's function for the operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \lambda^2, \quad 0 < x < 1,$$

i.e. find $G = G(x,\xi)$ such that for each $0 < \xi < 1$ we have $LG = \delta(x-\xi)$ and $G(0,\xi) = G(1,\xi) = 0$. Hence show that the solution to the equation $Ly = f$ with $y(0) = y(1) = 0$ is given by

$$y(x) = \frac{1}{\lambda \sinh \lambda}\left[\sinh(\lambda x)\int_x^1 f(\xi)\sinh\left[\lambda(1-\xi)\right]\mathrm{d}\xi + \sinh\left[\lambda(1-x)\right]\int_0^x f(\xi)\sinh(\lambda\xi)\,\mathrm{d}\xi\right].$$

Does this solution hold in the limit $\lambda \to 0$? Why does this formula break down when $\lambda = m\pi\mathrm{i}$, $m \in \mathbf{N}$? *Hint: when are there nontrivial solutions to $Ly = 0$ satisfying $y(0) = y(1) = 0$?*

5. The function $\theta = \theta(t)$ measures the displacement of a damped oscillator. It satisfies the initial value problem

$$\begin{cases} \ddot{\theta} + 2p\dot{\theta} + (p^2+q^2)\theta = f(t), \quad t > 0 \\ y(0) = \dot{y}(0) = 0, \end{cases}$$

where p, q are real constants with $p > 0$ and $q \neq 0$. By constructing an appropriate Green's function, show that

$$\theta(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)}\sin\left[q(t-\tau)\right] f(\tau)\,\mathrm{d}\tau.$$

Obtain the same result using the Fourier transform.

6. Calculate the Fourier transforms of the functions that are zero on $|x| > c$ and otherwise defined on $|x| \leq c$ by

$$\text{(i) } f(x) = 1, \qquad \text{(ii) } f(x) = e^{\mathrm{i}ax}, \qquad \text{(iii) } f(x) = \sin(ax), \qquad \text{(iv) } f(x) = \cos(ax).$$

7. Calculate the Fourier transforms of the following functions in terms of the Dirac delta function

$$\text{(i) } f(x) = 1, \qquad \text{(ii) } f(x) = e^{\mathrm{i}ax}, \qquad \text{(iii) } f(x) = \sin(ax), \qquad \text{(iv) } f(x) = \cos(ax).$$

Compare your answers to the previous question and comment on the results.

8. Compute the discrete Fourier transform of the sequence $X_n = n$ for $n = 0, 1, \ldots, N-1$. Hence show that

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{k=1}^{N-1} \frac{1}{\sin^2(\pi k/N)} = \frac{1}{3}.$$

9. By considering the Fourier transform of the function $f(x) = \cos(x)$ when $|x| < \pi/2$ and $f(x) = 0$ when $|x| \geq \pi/2$, and the Fourier transform of its derivative, show that

$$\int_0^\infty \frac{\cos^2(\pi t/2)}{(1-t^2)^2}\,\mathrm{d}t = \int_0^\infty \frac{t^2\cos^2(\pi t/2)}{(1-t^2)^2}\,\mathrm{d}t = \frac{\pi^2}{8}.$$

10. By choosing an appropriate function in the Poisson summation formula, establish the identity

$$\sqrt{\alpha}\left[\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\alpha^2 n^2/2}\right] = \sqrt{\beta}\left[\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\beta^2 n^2/2}\right] \quad \text{where } \alpha\beta = 2\pi.$$

Deduce that

$$\sum_{n=1}^{\infty} \left(4\pi n^2 - 1\right) e^{-\pi n^2} = \frac{1}{2}.$$

Additional problems

*These questions should **not** be attempted at the expense of earlier ones.*

11. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x^2+y^2\right)\delta'\left(x^2+y^2-1\right)\delta\left(x^2-y^2\right)\,\mathrm{d}x\,\mathrm{d}y = f(1) - f'(1)$$

in two different ways: (a) use the identity derived in question 2; and (b) using plane polar coordinates.

12. By constructing a suitable Green's function, find the solution to the initial value problem

$$\begin{cases} y^{(n)} = f, \quad t > 0 \\ y(0) = \cdots = y^{(n-1)}(0) = 0. \end{cases}$$

Hence prove Taylor's theorem with integral remainder for a n-times continuously differentiable functions.