

## METHODS — EXAMPLES IV

## General properties of PDEs

1. *Characteristics.*

(i) Find the characteristic curves of  $u_x + yu_y = 0$ . Hence find the solution of the problem with the boundary data  $u(0, y) = y^3$ .

(ii) Solve for  $u$  which satisfies  $yu_x + xu_y = 0$  with  $u(0, y) = e^{-y^2}$ . In which region of the plane is the solution uniquely determined?

(iii) Find  $u$  such that  $u_x + u_y + u = e^{x+2y}$ , and  $u(x, 0) = 0$ .

2. *Well-posedness.*

The **backward** diffusion equation may be defined as

$$u_{xx} + u_t = 0.$$

Consider a domain  $0 < x < \pi$ , with  $u(0, t) = 0 = u(\pi, t)$ , and  $u(x, 0) = U(x)$ . Use the method of separation of variables to show that the solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{n^2 t}, \quad \text{where } b_n = \frac{2}{\pi} \int_0^{\pi} U(x) \sin(nx) dx.$$

Hence demonstrate that this problem is not well-posed.

3. *Classification.*

(i) Determine the regions where Tricomi's equation

$$u_{xx} + xu_{yy} = 0,$$

is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.

(ii) Reduce the equation

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0,$$

to the simple canonical form  $u_{\xi\eta} = 0$  in its hyperbolic region, and hence show that

$$u = f(x + 2[-y]^{1/2}) + g(x - 2[-y]^{1/2}),$$

where  $f$  and  $g$  are arbitrary functions.

## Applications of Green's functions

4. *Cauchy problem in the half-plane for the Laplacian.* Consider Laplace's equation in the half-plane with prescribed boundary conditions at  $y = 0$ , i.e.

$$\nabla^2 \psi = 0; \quad -\infty < x < \infty, \quad y \geq 0,$$

where  $\psi(x, 0) = f(x)$  a known function, such that  $\psi$  tends to zero as  $y \rightarrow \infty$ .

(i) Find the Green's function for this problem.

(ii) Hence show that the solution is given by (another!) Poisson's integral formula:

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi.$$

(iii) Derive the same result by taking Fourier transforms with respect to  $x$  (assuming all transforms exist).

(iv) Find (in closed form) the solution when  $f(x) = \psi_0$ ,  $|x| < a$ , and  $f(x) = 0$  otherwise. Sketch this solution (a) for various  $y > 0$  and (b) along  $x = \pm a$ .

5. *Diffusion equation with a boundary source.* Consider the problem on the half-line:

$$\theta_t - D\theta_{xx} = f(x, t), \quad 0 < x < \infty, \quad 0 < t < \infty,$$

with boundary and initial data  $\theta(0, t) = h(t)$ ,  $\theta(x, 0) = \Theta(x)$ . By considering the variable  $V(x, t) = \theta(x, t) - h(t)$ , and using the method of images, derive the general solution.

**6. Forced wave equation.**

An infinite string, at rest for  $t < 0$ , receives an instantaneous transverse blow at  $t = 0$  which imparts an initial velocity of  $V\delta(x - x_0)$ , where  $V$  is a constant. Derive the position of the string for  $t > 0$ .

**7. Forced wave equation: Method of images.**

A semi-infinite string, fixed for all time at zero at  $x = 0$  and at rest for  $t < 0$ , receives an instantaneous transverse blow at  $t = 0$  which imparts an initial velocity of  $V\delta(x - x_0)$ , where  $V$  is a constant. Derive the position of the string for  $t > 0$ , and compare the solution to the infinite case in the previous question.

**8. Dirichlet Green's function for the sphere\*.**

(i) Verify that the Dirichlet Green's function for the Laplacian for the **interior** of a spherical domain of radius  $a$  is

$$G(\mathbf{x}; \mathbf{x}_0) = \frac{-1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{a}{|\mathbf{x}_0|} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0^*|}, \quad \mathbf{x}_0^* = \frac{a^2\mathbf{x}_0}{|\mathbf{x}_0|^2}.$$

(ii) Find the Dirichlet Green's function for the Laplacian for the **exterior** of a spherical domain of radius  $a$ .

### Properties of Green's functions

**9. Representation formula in 2D.** If  $u$  is a harmonic function in a 2D domain  $\mathcal{D}$ , with boundary  $\delta\mathcal{D}$ , show that

$$u(\mathbf{x}_0) = \frac{1}{2\pi} \oint_{\delta\mathcal{D}} \left[ u(\mathbf{x}) \frac{\partial}{\partial n} (\log |\mathbf{x} - \mathbf{x}_0|) - \log |\mathbf{x} - \mathbf{x}_0| \frac{\partial u}{\partial n} \right] dl,$$

where  $dl$  is an arc element of  $\delta\mathcal{D}$ ,  $\mathbf{x} \in \delta\mathcal{D}$ ,  $\mathbf{x}_0 \in \mathcal{D}$ .

**10. Application of boundary conditions.**

Consider the problem

$$\nabla^2 u = 0, \quad u(x, y, 0) = h(x, y), \quad u \rightarrow 0 \text{ as } x^2 + y^2 \rightarrow \infty,$$

where  $h(x, y)$  is bounded and continuous, which has the solution

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{-3/2} h(x, y) dx dy.$$

Verify directly from the formula that the boundary conditions are satisfied.

[Hint: It may be helpful to change variables to  $z_0^2 s^2 = (x - x_0)^2 + (y - y_0)^2$ .]

**11. Symmetry.**

Consider a Green's function  $G(\mathbf{r}_1; \mathbf{r}_2)$  for the Laplacian defined in an arbitrary three-dimensional domain  $\mathcal{D}$ . By using Green's second identity, show that  $G(\mathbf{r}_1; \mathbf{r}_2) = G(\mathbf{r}_2; \mathbf{r}_1)$  for all  $\mathbf{r}_1 \neq \mathbf{r}_2$  in the domain  $\mathcal{D}$ .

### Bessel functions revisited

**12. Laplacian in cylindrical polar coordinates.** Consider the problem  $\nabla^2 u = 0$ ,  $r \neq 0$ ,  $u \rightarrow 0$  as  $r \rightarrow \infty$ . Show that a solution of this equation which is independent of polar angle is  $u_1 = 1/r = 1/(R^2 + z^2)^{1/2}$  where  $R$  is the radial component in cylindrical polar coordinates. By considering the Laplacian in cylindrical polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

and separating variables, show that, for an arbitrary function  $f(\lambda)$ ,

$$u_2 = \int_0^\infty f(\lambda) e^{-\lambda|z|} J_0(\lambda R) d\lambda,$$

is harmonic for  $z > 0$  and  $z < 0$ . Now let  $f(\lambda) = 1$ . Assuming that  $\int_0^\infty J_0(z) dz = 1$ , show that  $u_2 = 1/R$  on the  $z = 0$  plane. Explain why it is plausible that  $u_2 = 1/r$  everywhere (you need not prove this) and deduce, if so, that

$$\int_0^\infty e^{-\lambda|z|} J_0(\lambda R) d\lambda = \frac{1}{\sqrt{R^2 + z^2}}.$$

This is effectively a derivation of the **Laplace transform** of  $J_0(\lambda R)$ .

<sup>†</sup>If you find any errors in the Methods Examples sheets, please inform your supervisor or email [epss@damtp.cam.ac.uk](mailto:epss@damtp.cam.ac.uk).