Mathematical Tripos Part IB  
Michaelmas term 2020  
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METHODS — EXAMPLES IV

General properties of PDEs

1. Characteristics.
   (i) Find the characteristic curves of \( u_x + yu_y = 0 \). Hence find the solution of the problem with the boundary data \( u(0, y) = y^3 \).
   (ii) Solve for \( u \) which satisfies \( yu_x + xu_y = 0 \) with \( u(0, y) = e^{-y^2} \). In which region of the plane is the solution uniquely determined?
   (iii) Find \( u \) such that \( u_x + u_y + u = e^{x+2y} \), and \( u(x, 0) = 0 \).

2. Well-posedness.
   The backward diffusion equation may be defined as \( u_{xx} + u_t = 0 \).
   Consider a domain \( 0 < x < \pi \), with \( u(0, t) = 0 = u(\pi, t) \), and \( u(x, 0) = U(x) \). Use the method of separation of variables to show that the solution is
   \[
   u(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{n^2 t},
   \]
   where
   \[
   b_n = \frac{2}{\pi} \int_0^\pi U(x) \sin(nx) dx.
   \]
   Hence demonstrate that this problem is not well-posed.

3. Classification.
   (i) Determine the regions where Tricomi’s equation \( u_{xx} + xu_{yy} = 0 \) is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.
   (ii) Reduce the equation \( u_{xx} + yu_{yy} + \frac{1}{2} u_y = 0 \) to the simple canonical form \( u_{t\eta} = 0 \) in its hyperbolic region, and hence show that
   \[
   u = f(x + 2[-y]^{1/2}) + g(x - 2[-y]^{1/2}),
   \]
   where \( f \) and \( g \) are arbitrary functions.

Applications of Green’s functions

4. Cauchy problem in the half-plane for the Laplacian. Consider Laplace’s equation in the half-plane with prescribed boundary conditions at \( y = 0 \), i.e.
   \[
   \nabla^2 \psi = 0; \ -\infty < x < \infty, \ y \geq 0,
   \]
   where \( \psi(x, 0) = f(x) \) a known function, such that \( \psi \) tends to zero as \( y \to \infty \).
   (i) Find the Green’s function for this problem.
   (ii) Hence show that the solution is given by (another!) Poisson’s integral formula:
   \[
   \psi(x, y) = \frac{y}{\pi} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi;
   \]
   (iii) Derive the same result by taking Fourier transforms with respect to \( x \) (assuming all transforms exist).
   (iv) Find (in closed form) the solution when \( f(x) = \psi_0, |x| < a \), and \( f(x) = 0 \) otherwise. Sketch this solution (a) for various \( y > 0 \) and (b) along \( x = \pm a \).

5. Diffusion equation with a boundary source. Consider the problem on the half-line:
   \[
   \theta_t - \theta_{xx} = f(x, t), 0 < x < \infty, 0 < t < \infty,
   \]
   with boundary and initial data \( \theta(0, t) = h(t), \theta(x, 0) = \Theta(x) \). By considering the variable \( V(x, t) = \theta(x, t) - h(t) \), and using the method of images, derive the general solution.
6. **Forced wave equation.**
An infinite string, at rest for \( t < 0 \), receives an instantaneous transverse blow at \( t = 0 \) which imparts an initial velocity of \( V \delta(x - x_0) \), where \( V \) is a constant. Derive the position of the string for \( t > 0 \).

7. **Forced wave equation: Method of images.**
A semi-infinite string, fixed for all time at zero at \( x = 0 \) and at rest for \( t < 0 \), receives an instantaneous transverse blow at \( t = 0 \) which imparts an initial velocity of \( V \delta(x - x_0) \), where \( V \) is a constant. Derive the position of the string for \( t > 0 \), and compare the solution to the infinite case in the previous question.

8. **Dirichlet Green’s function for the sphere.**
(i) Verify that the Dirichlet Green’s function for the Laplacian for the interior of a spherical domain of radius \( a \) is

\[
G(x; x_0) = \frac{-1}{4\pi|x - x_0|} + \frac{a}{4\pi|x - x_0|^2}, \quad x_0 = \frac{a^2|x_0|}{|x_0|^2}.
\]

(ii) Find the Dirichlet Green’s function for the Laplacian for the exterior of a spherical domain of radius \( a \).

**Properties of Green’s functions.**

9. **Representation formula in 2D.**
If \( \alpha \) is a harmonic function in a 2D domain \( D \), with boundary \( \partial D \), show that

\[
u(x_0) = \frac{1}{2\pi} \int_{\partial D} \left[ u(x) \frac{\partial}{\partial n} (\log |x - x_0|) - \log |x - x_0| \frac{\partial u}{\partial n} \right] dl,
\]

where \( dl \) is an arc element of \( \partial D \), \( x \in \partial D \), \( x_0 \in D \).

10. **Application of boundary conditions.**
Consider the problem

\[
\nabla^2 u = 0, \quad u(x, y, 0) = h(x, y), \quad u \to 0 \text{ as } x^2 + y^2 \to \infty,
\]

where \( h(x, y) \) is bounded and continuous, which has the solution

\[
u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (x - x_0)^2 + (y - y_0)^2 + z_0^2 \right]^{-3/2} h(x, y) dx dy.
\]

Verify directly from the formula that the boundary conditions are satisfied.

[Hint: It may be helpful to change variables to \( z_0^2 s^2 = (x - x_0)^2 + (y - y_0)^2 \).]

11. **Symmetry.**
Consider a Green’s function \( G(r_1; r_2) \) for the Laplacian defined in an arbitrary three-dimensional domain \( D \). By using Green’s second identity, show that \( G(r_1; r_2) = G(r_2; r_1) \) for all \( r_1 \neq r_2 \) in the domain \( D \).

**Bessel functions revisited.**

12. **Laplacian in cylindrical polar coordinates.**
Consider the problem \( \nabla^2 u = 0, \ r \neq 0, \ u \to 0 \text{ as } r \to \infty \). Show that a solution of this equation which is independent of polar angle is \( u_1 = 1/r = 1/(R^2 + z^2)^{1/2} \) where \( R \) is the radial component in cylindrical polar coordinates. By considering the Laplacian in cylindrical polar coordinates

\[
\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},
\]

and separating variables, show that, for an arbitrary function \( f(\lambda) \),

\[
u_2 = \int_0^\infty f(\lambda)e^{-\lambda|z|}J_0(\lambda R) d\lambda,
\]

is harmonic for \( z > 0 \) and \( z < 0 \). Now let \( f(\lambda) = 1 \). Assuming that \( \int_0^\infty J_0(\lambda z)dz = 1 \), show that \( u_2 = 1/R \) on the \( z = 0 \) plane. Explain why it is plausible that \( u_2 = 1/r \) everywhere (you need not prove this) and deduce, if so, that

\[
\int_0^\infty e^{-\lambda|z|}J_0(\lambda R) d\lambda = \frac{1}{\sqrt{R^2 + z^2}}.
\]

This is effectively a derivation of the **Laplace transform** of \( J_0(\lambda R) \).

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1. If you find any errors in the Methods Examples sheets, please inform your supervisor or email epss@damtp.cam.ac.uk.