General properties of PDEs

1. **Characteristics.**
   i) Find the characteristic curves of \( u_x + yu_y = 0 \). Hence find the solution of the problem with the boundary data \( u(0, y) = y^3 \).
   ii) Solve for \( u \) which satisfies \( yu_x + xu_y = 0 \) with \( u(0, y) = e^{-y^2} \). In which region of the plane is the solution uniquely determined?
   iii) Find \( u \) such that \( u_x + u_y + u = e^{x+2y} \), and \( u(x, 0) = 0 \).

2. **Well-posedness.**
   The backward diffusion equation may be defined as \( u_{xx} + u_t = 0 \).

   Consider a domain \( 0 < x < \pi \), with \( u(0, t) = 0 = u(\pi, t) \), and \( u(x, 0) = U(x) \). By using the method of separation of variables, show that the problem is not well-posed. [It may be helpful to scale the eigenfunctions you calculate similarly to the example in the lectures.]

3. **Classification.**
   i) Determine the regions where Tricomi’s equation \( u_{xx} + xu_{yy} = 0 \) is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.
   ii) Reduce the equation \( u_{xx} + yu_{yy} + \frac{1}{2} u_y = 0 \) to the simple canonical form \( u_{\xi \eta} = 0 \) in its hyperbolic region, and hence show that
   \[
   u = f(x + 2[-y]^{1/2}) + g(x - 2[-y]^{1/2}),
   \]
   where \( f \) and \( g \) are arbitrary functions.

Properties of Green’s functions

4. **Symmetry.**
   Consider a Dirichlet Green’s function \( G(r; r_0) \) for the Laplacian defined in an arbitrary three-dimensional domain \( D \). By using Green’s second identity, show that \( G(r; r_0) = G(r_0; r) \) for all \( r \neq r_0 \) in the domain \( D \).

5. **Representation formula in 2D.**
   If \( u \) is a harmonic function in a 2D domain \( D \), with boundary \( \delta D \), show that
   \[
   u(x_0) = \frac{1}{2\pi} \oint_{\delta D} \left[ u(x) \left( \frac{\partial}{\partial n} \left( \log |x - x_0| \right) - \log |x - x_0| \frac{\partial u}{\partial n} \right) \right] dl,
   \]
   where \( dl \) is an arc element of \( \delta D \), \( x \in \delta D \), \( x_0 \in D \).

Applications of Green’s functions

6. **Cauchy problem in the half-plane for the Laplacian.**
   Consider Laplace’s equation in the half-plane with prescribed boundary conditions at \( y = 0 \), i.e.
   \[
   \nabla^2 \psi = 0; \quad -\infty < x < \infty, \quad y \geq 0,
   \]
   where \( \psi(x, 0) = f(x) \) a known function, such that \( \psi \) tends to zero as \( y \to \infty \).
   i) Find the Green’s function for this problem.
   ii) Hence show that the solution is given by Poisson's integral formula:
   \[
   \psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi.
   \]
iii) Derive the same result by taking Fourier transforms with respect to $x$ (assuming all transforms exist).

iv) Find (in closed form) and sketch the solution for various $y > 0$ when $f(x) = \psi_0$, $|x| < a$, and $f(x) = 0$ otherwise. Sketch the solution along $x = \pm a$.

7. Wave equation.
An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V \delta(x - x_0)$, where $V$ is a constant. Derive the position of the string for $t > 0$.

8. Wave equation: Method of images.
A semi-infinite string, fixed for all time at zero at $x = 0$ and at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V \delta(x - x_0)$, where $V$ is a constant. Derive the position of the string for $t > 0$, and compare the solution to the infinite case in the previous question.

9. Diffusion equation with a boundary source.
Consider the problem on the half-line:

$$\partial_t \theta - D \partial_{xx} \theta = f(x, t), \quad 0 < x < \infty, \quad 0 < t < \infty,$$

with boundary and initial data $\theta(0, t) = h(t)$, $\theta(x, 0) = \Theta(x)$. By considering the variable $V(x, t) = \theta(x, t) - h(t)$, and using the method of images, derive the general solution.

10. Dirichlet Green’s function for the sphere.

i) Show that the Dirichlet Green’s function for the Laplacian for the interior of a spherical domain of radius $a$ is

$$G(x; x_0) = \frac{-1}{4\pi|x - x_0|} + \frac{a}{4\pi|x - x_0^*|}, \quad \text{where} \quad x_0^* = \frac{a^2 x_0}{|x_0|^2}.$$

ii) Derive the Dirichlet Green’s function for the Laplacian for the exterior of a spherical domain of radius $a$.

11. Forced wave equation.

Consider the forced wave equation with zero initial conditions

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x, t) \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0.$$

Verify directly that

$$u(x, t) = \frac{1}{2c} \int_0^t \int_{x - c(t-s)}^{x + c(t-s)} f(y, s) \, dy \, ds,$$

and hence determine the appropriate Green’s function for the wave equation satisfying

$$\frac{\partial^2}{\partial t^2} G(x, t; \xi, \tau) - c^2 \frac{\partial^2}{\partial x^2} G(x, t; \xi, \tau) = \delta(x - \xi) \delta(t - \tau),$$

$$G(x, 0; \xi, \tau) = 0, \quad \frac{\partial}{\partial t} G(x, 0; \xi, \tau) = 0.$$

Calculate $u(x, t)$ explicitly in the case where $f(x, t) = \cos x$ and hence determine the times when $u = 0$ for all values of $x$. 