METHODS — EXAMPLES IV

General properties of PDEs

1. *Characteristics.*
(i) Find the characteristic curves of $u_x + yu_y = 0$. Hence find the solution of the problem with the boundary data $u(0, y) = y^3$.

(ii) Solve for u which satisfies $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. In which region of the plane is the solution uniquely determined?

(iii) Find u such that $u_x + u_y + u = e^{x+2y}$, and $u(x, 0) = 0$.

2. *Well-posedness.*
The **backward** diffusion equation may be defined as

$$u_{xx} + u_t = 0.$$

Consider a domain $0 < x < \pi$, with $u(0, t) = 0 = u(\pi, t)$, and $u(x, 0) = U(x)$. Use the method of separation of variables to show that the solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{n^2 t}, \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} U(x) \sin(nx) dx.$$

Hence demonstrate that this problem is not well-posed.

3. *Classification.*
(i) Determine the regions where Tricomi's equation

$$u_{xx} + xu_{yy} = 0,$$

is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.

(ii) Reduce the equation

$$u_{xx} + yu_{yy} + \frac{1}{2} u_y = 0,$$

to the simple canonical form $u_{\xi\eta} = 0$ in its hyperbolic region, and hence show that

$$u = f(x + 2[-y]^{1/2}) + g(x - 2[-y]^{1/2}),$$

where f and g are arbitrary functions.

Applications of Green's functions

4. *Cauchy problem in the half-plane for the Laplacian.* Consider Laplace's equation in the half-plane with prescribed boundary conditions at $y = 0$, i.e.

$$\nabla^2 \psi = 0; \ -\infty < x < \infty, \ y \geq 0,$$

where $\psi(x, 0) = f(x)$ a known function, such that ψ tends to zero as $y \to \infty$.

(i) Find the Green's function for this problem.

(ii) Hence show that the solution is given by (another!) Poisson's integral formula:

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi.$$

(iii) Derive the same result by taking Fourier transforms with respect to x (assuming all transforms exist).

(iv) Find (in closed form) the solution when $f(x) = \psi_0$, $|x| < a$, and $f(x) = 0$ otherwise. Sketch this solution (a) for various $y > 0$ and (b) along $x = \pm a$.

5. *Diffusion equation with a boundary source.* Consider the problem on the half-line:

$$\theta_t - D\theta_{xx} = f(x, t), 0 < x < \infty, \ 0 < t < \infty,$$

with boundary and initial data $\theta(0, t) = h(t)$, $\theta(x, 0) = \Theta(x)$. By considering the variable $V(x, t) = \theta(x, t) - h(t)$, and using the method of images, derive the general solution.

6. *Forced wave equation.*
An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where V is a constant. Derive the position of the string for $t > 0$.

7. *Forced wave equation: Method of images.*
A semi-infinite string, fixed for all time at zero at $x = 0$ and at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where V is a constant. Derive the position of the string for $t > 0$, and compare the solution to the infinite case in the previous question.

8. *Dirichlet Green's function for the sphere*.*
(i) Verify that the Dirichlet Green's function for the Laplacian for the **interior** of a spherical domain of radius a is

$$G(\mathbf{x};\mathbf{x}_0) = \frac{-1}{4\pi|\mathbf{x}-\mathbf{x}_0|} + \frac{a}{|\mathbf{x}_0|}\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0^*|}, \qquad \mathbf{x}_0^* = \frac{a^2\mathbf{x}_0}{|\mathbf{x}_0|^2}.$$

(ii) Find the Dirichlet Green's function for the Laplacian for the **exterior** of a spherical domain of radius a.

Properties of Green's functions

9. *Representation formula in 2D.* If u is a harmonic function in a 2D domain $\mathcal{D}$, with boundary $\delta\mathcal{D}$, show that

$$u(\mathbf{x}_0) = \frac{1}{2\pi}\oint_{\delta\mathcal{D}}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\log|\mathbf{x}-\mathbf{x}_0|) - \log|\mathbf{x}-\mathbf{x}_0|\frac{\partial u}{\partial n}\right]dl,$$

where dl is an arc element of $\delta\mathcal{D}$, $\mathbf{x} \in \delta\mathcal{D}$, $\mathbf{x}_0 \in \mathcal{D}$.

10. *Application of boundary conditions.*
Consider the problem

$$\nabla^2 u = 0,\ u(x,y,0) = h(x,y), \quad u \to 0 \text{ as } x^2+y^2 \to \infty,$$

where $h(x,y)$ is bounded and continuous, which has the solution

$$u(x_0,y_0,z_0) = \frac{z_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(x-x_0)^2+(y-y_0)^2+z_0^2\right]^{-3/2} h(x,y)dxdy.$$

Verify directly from the formula that the boundary conditions are satisfied.
[*Hint: It may be helpful to change variables to* $z_0^2 s^2 = (x-x_0)^2+(y-y_0)^2$.]

11. *Symmetry.*
Consider a Green's function $G(\mathbf{r}_1;\mathbf{r}_2)$ for the Laplacian defined in an arbitrary three-dimensional domain $\mathcal{D}$. By using Green's second identity, show that $G(\mathbf{r}_1;\mathbf{r}_2) = G(\mathbf{r}_2;\mathbf{r}_1)$ for all $\mathbf{r}_1 \neq \mathbf{r}_2$ in the domain $\mathcal{D}$.

Bessel functions revisited

12. *Laplacian in cylindrical polar coordinates.* Consider the problem $\nabla^2 u = 0$, $r \neq 0$, $u \to 0$ as $r \to \infty$. Show that a solution of this equation which is independent of polar angle is $u_1 = 1/r = 1/(R^2+z^2)^{1/2}$ where R is the radial component in cylindrical polar coordinates. By considering the Laplacian in cylindrical polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2},$$

and separating variables, show that, for an arbitrary function $f(\lambda)$,

$$u_2 = \int_0^\infty f(\lambda)e^{-\lambda|z|}J_0(\lambda R)d\lambda,$$

is harmonic for $z > 0$ and $z < 0$. Now let $f(\lambda) = 1$. Assuming that $\int_0^\infty J_0(z)\,dz = 1$, show that $u_2 = 1/R$ on the $z = 0$ plane. Explain why it is plausible that $u_2 = 1/r$ everywhere (you need not prove this) and deduce, if so, that

$$\int_0^\infty e^{-\lambda|z|}J_0(\lambda R)d\lambda = \frac{1}{\sqrt{R^2+z^2}}.$$

This is effectively a derivation of the **Laplace transform** of $J_0(\lambda R)$.

†If you find any errors in the Methods Examples sheets, please inform your supervisor or email epss@damtp.cam.ac.uk.