**METHODS — EXAMPLES IV**

**General properties of PDEs**

1. **Characteristics.**
   i) Find the characteristic curves of \( u_x + yu_y = 0 \). Hence find the solution of the problem with the boundary data \( u(0, y) = y^3 \).
   ii) Solve for \( u \) which satisfies \( yu_x + xu_y = 0 \) with \( u(0, y) = e^{-y^3} \). In which region of the plane is the solution uniquely determined?
   iii) Find \( u \) such that \( u_x + u_y + u = e^{x+2y} \), and \( u(x, 0) = 0 \).

2. **Well-posedness.**
   The backward diffusion equation may be defined as \( u_{xx} + u_t = 0 \).
   Consider a domain \( 0 < x < \pi \), with \( u(0, t) = 0 = u(\pi, t) \), and \( u(x, 0) = U(x) \). Use the method of separation of variables to show that the solution is
   \[
   u(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{n^2 t},
   \]
   where \( b_n = \frac{2}{\pi} \int_{0}^{\pi} U(x) \sin(nx) dx \).
   Hence demonstrate that this problem is not well-posed.

3. **Classification.**
   i) Determine the regions where Tricomi’s equation \( u_{xx} + xu_{yy} = 0 \) is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.
   ii) Reduce the equation \( u_{xx} + yu_{yy} + \frac{1}{2} u_y = 0 \), to the simple canonical form \( u_{\xi \eta} = 0 \) in its hyperbolic region, and hence show that
   \[
   u = f(x + 2[-y]^{1/2}) + g(x - 2[-y]^{1/2}),
   \]
   where \( f \) and \( g \) are arbitrary functions.

**Applications of Green’s functions**

4. **Cauchy problem in the half-plane for the Laplacian.** Consider Laplace’s equation in the half-plane with prescribed boundary conditions at \( y = 0 \), i.e.
   \[
   \nabla^2 \psi = 0; \quad -\infty < x < \infty, \quad y \geq 0,
   \]
   where \( \psi(x, 0) = f(x) \) a known function, such that \( \psi \) tends to zero as \( y \to \infty \).
   i) Find the Green’s function for this problem.
   ii) Hence show that the solution is given by (another!) Poisson’s integral formula:
   \[
   \psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2 + y^2} d\xi.
   \]
   iii) Derive the same result by taking Fourier transforms with respect to \( x \) (assuming all transforms exist).
   iv) Find (in closed form) and sketch the solution for various \( y > 0 \) when \( f(x) = \psi_0, \ |x| < a \), and \( f(x) = 0 \) otherwise. Sketch the solution along \( x = \pm a \).
   v) Calculate the solution when \( f(x) = \psi_0 \) for all \( x \).

5. **Diffusion equation with a boundary source.** Consider the problem on the half-line:
   \[
   \theta_t - D\theta_{xx} = f(x, t), \quad 0 < x < \infty, \quad 0 < t < \infty,
   \]
   with boundary and initial data \( \theta(0, t) = h(t), \ \theta(x, 0) = \Theta(x) \). By considering the variable \( V(x, t) = \theta(x, t) - h(t) \), and using the method of images, derive the general solution.
6. **Forced wave equation.** An infinite string, at rest for \(t < 0\), receives an instantaneous transverse blow at \(t = 0\) which imparts an initial velocity of \(V\delta(x - x_0)\), where \(V\) is a constant. Derive the position of the string for \(t > 0\).

7. **Forced wave equation: Method of images.** A semi-infinite string, fixed for all time at \(x = 0\) and at rest for \(t < 0\), receives an instantaneous transverse blow at \(t = 0\) which imparts an initial velocity of \(V\delta(x - x_0)\), where \(V\) is a constant. Derive the position of the string for \(t > 0\), and compare the solution to the infinite case in the previous question.

8. **Dirichlet Green’s function for the sphere**.
   i) Show that the Dirichlet Green’s function for the Laplacian for the interior of a spherical domain of radius \(a\) is
   \[
   G(x; x_0) = \frac{-1}{4\pi|x - x_0|} + \frac{a}{|x_0| 4\pi|x - x_0|^2} \quad x_0 = \frac{a^2 x_0}{|x_0|^2}.
   \]
   ii) Derive the Dirichlet Green’s function for the Laplacian for the exterior of a spherical domain of radius \(a\).

**Properties of Green’s functions**

9. **Representation formula in 2D.** If \(u\) is a harmonic function in a 2D domain \(D\), with boundary \(\delta D\), show that
   \[
   u(x_0) = \frac{1}{2\pi} \int_{\delta D} \left[ u(x) \frac{\partial}{\partial n} (\log |x - x_0|) - \log |x - x_0| \frac{\partial u}{\partial n} \right] \, dl,
   \]
   where \(dl\) is an arc element of \(\delta D\), \(x \in \delta D\), \(x_0 \in D\).

10. **Application of boundary conditions.** Consider the problem
    \[
    \nabla^2 u = 0, \quad u(x, y, 0) = h(x, y), \quad u \to 0 \text{ as } x^2 + y^2 \to \infty, \quad h(x, y) \text{ bounded and continuous},
    \]
    which has solution
    \[
    u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} [(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{-3/2} h(x, y) \, dx \, dy.
    \]
    Verify directly from the formula that the boundary conditions are satisfied. [Changing variables to \(z_0^2 s^2 = (x - x_0)^2 + (y - y_0)^2 \) may be helpful.]

11. **Symmetry.** Consider a Green’s function \(G(r_1; r_2)\) for the Laplacian defined in an arbitrary three-dimensional domain \(D\). By using Green’s second identity, show that \(G(r_1; r_2) = G(r_2; r_1)\) for all \(r_1 \neq r_2\) in the domain \(D\).

**Bessel functions revisited**

12. **Laplacian in cylindrical polar coordinates.** Consider the problem \(\nabla^2 u = 0, \quad r \neq 0, \quad u \to 0 \text{ as } r \to \infty\). Show that a solution of this equation which is independent of polar angle is \(u_1 = 1/r = 1/(R^2 + z^2)^{1/2}\) where \(R\) is the radial component in cylindrical polar coordinates. By considering the Laplacian in cylindrical polar coordinates
    \[
    \nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},
    \]
    and separating variables, show that, for an arbitrary function \(f(\lambda),
    \[
    u_2 = \int_0^\infty f(\lambda) e^{-\lambda|z|} J_0(\lambda R) d\lambda,
    \]
    is also a solution which is independent of polar angle. By requiring \(u_2 = u_1\), and then comparing these solutions on the axis \(R = 0\), show that \(f(\lambda) = 1\) is an admissible choice for \(f(\lambda)\) and hence that
    \[
    \int_0^\infty e^{-\lambda|z|} J_0(\lambda R) d\lambda = \frac{1}{\sqrt{R^2 + z^2}}.
    \]
    This is effectively a derivation of the **Laplace transform** of \(J_0(\lambda R)\).

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¹If you find any errors in the Methods Examples sheets, please inform your supervisor or email c.p.caulfield@bpi.cam.ac.uk.