METHODS — EXAMPLES IV

General properties of PDEs

1. *Characteristics*.
i) Find the characteristic curves of $u_x + yu_y = 0$. Hence find the solution of the problem with the boundary data $u(0, y) = y^3$.
ii) Solve for u which satisfies $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. In which region of the plane is the solution uniquely determined?
iii) Find u such that $u_x + u_y + u = e^{x+2y}$, and $u(x, 0) = 0$.

2. *Well-posedness*.
The **backward** diffusion equation may be defined as

$$u_{xx} + u_t = 0.$$

Consider a domain $0 < x < \pi$, with $u(0, t) = 0 = u(\pi, t)$, and $u(x, 0) = U(x)$. By using the method of separation of variables, show that the problem is not well-posed. [*It may be helpful to scale the eigenfunctions you calculate similarly to the example in the lectures.*]

3. *Classification*.
i) Determine the regions where Tricomi's equation

$$u_{xx} + xu_{yy} = 0,$$

is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.
ii) Reduce the equation

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0,$$

to the simple canonical form $u_{\xi\eta} = 0$ in its hyperbolic region, and hence show that

$$u = f(x + 2[-y]^{1/2}) + g(x - 2[-y]^{1/2}),$$

where f and g are arbitrary functions.

Properties of Green's functions

4. *Symmetry*.
Consider a Dirichlet Green's function $G(\mathbf{r}; \mathbf{r}_0)$ for the Laplacian defined in an arbitrary three-dimensional domain $\mathcal{D}$. By using Green's second identity, show that $G(\mathbf{r}; \mathbf{r}_0) = G(\mathbf{r}_0; \mathbf{r})$ for all $\mathbf{r} \neq \mathbf{r}_0$ in the domain $\mathcal{D}$.

5. *Representation formula in 2D*.
If u is a harmonic function in a 2D domain $\mathcal{D}$, with boundary $\delta\mathcal{D}$, show that

$$u(\mathbf{x}_0) = \frac{1}{2\pi} \oint_{\delta\mathcal{D}} \left[u(\mathbf{x}) \frac{\partial}{\partial n} (\log |\mathbf{x} - \mathbf{x}_0|) - \log |\mathbf{x} - \mathbf{x}_0| \frac{\partial u}{\partial n} \right] dl,$$

where dl is an arc element of $\delta\mathcal{D}$, $\mathbf{x} \in \delta\mathcal{D}$, $\mathbf{x}_0 \in \mathcal{D}$.

Applications of Green's functions

6. *Cauchy problem in the half-plane for the Laplacian*.
Consider Laplace's equation in the half-plane with prescribed boundary conditions at $y = 0$, i.e.

$$\nabla^2\psi = 0; \ -\infty < x < \infty, \ y \geq 0,$$

where $\psi(x, 0) = f(x)$ a known function, such that ψ tends to zero as $y \to \infty$.
i) Find the Green's function for this problem.
ii) Hence show that the solution is given by Poisson's integral formula:

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi.$$

iii) Derive the same result by taking Fourier transforms with respect to x (assuming all transforms exist).
iv) Find (in closed form) and sketch the solution for various $y > 0$ when $f(x) = \psi_0$, $|x| < a$, and $f(x) = 0$ otherwise. Sketch the solution along $x = \pm a$.

7. Wave equation.

An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where V is a constant. Derive the position of the string for $t > 0$.

8. Wave equation: Method of images.

A semi-infinite string, fixed for all time at zero at $x = 0$ and at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where V is a constant. Derive the position of the string for $t > 0$, and compare the solution to the infinite case in the previous question.

9. Diffusion equation with a boundary source.

Consider the problem on the half-line:

$$\theta_t - D\theta_{xx} = f(x,t), \qquad 0 < x < \infty, \qquad 0 < t < \infty,$$

with boundary and initial data $\theta(0,t) = h(t)$, $\theta(x,0) = \Theta(x)$. By considering the variable $V(x,t) = \theta(x,t) - h(t)$, and using the method of images, derive the general solution.

10. Dirichlet Green's function for the sphere*.

i) Show that the Dirichlet Green's function for the Laplacian for the **interior** of a spherical domain of radius a is

$$G(\mathbf{x};\mathbf{x}_0) = \frac{-1}{4\pi|\mathbf{x}-\mathbf{x}_0|} + \frac{a}{|\mathbf{x}_0|}\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0^*|}, \qquad \text{where} \qquad \mathbf{x}_0^* = \frac{a^2\mathbf{x}_0}{|\mathbf{x}_0|^2}.$$

ii) Derive the Dirichlet Green's function for the Laplacian for the **exterior** of a spherical domain of radius a.

11. Forced wave equation.

Consider the forced wave equation with zero initial conditions

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Verify directly that

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds\,,$$

and hence determine the appropriate Green's function for the wave equation satisfying

$$\frac{\partial^2}{\partial t^2}G(x,t;\xi,\tau) - c^2\frac{\partial^2}{\partial x^2}G(x,t;\xi,\tau) = \delta(x-\xi)\,\delta(t-\tau),$$

$$G(x,0;\xi,\tau) = 0, \qquad \frac{\partial}{\partial t}G(x,0;\xi,\tau) = 0.$$

Calculate $u(x,t)$ explicitly in the case where $f(x,t) = \cos x$ and hence determine the times when $u = 0$ for all values of x.